Interaction of Composite Fermions with a Gauge Field in the Fractional Quantum Hall Regime

by

Yong Baek Kim

M.Sc., Pohang University of Science and Technology
B.Sc., Seoul National University

Submitted to the Department of Physics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

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Abstract

This thesis studies the problem of two-dimensional fermions interacting with a gauge field. This problem arises in a theory of the half-filled Landau level in connection with the composite fermion theory of the fractional quantum Hall effect. A composite fermion is generated by attaching even number of flux quanta to an electron. The transformation from the electron to the composite fermion can be realized by introducing an appropriate Chern-Simons gauge field. Especially, at the filling fraction $\nu = 1/2$, composite fermions see effectively zero magnetic field at the mean field level because of the cancellation between the average of the Chern-Simons gauge field (from the attached magnetic flux quanta) and the external magnetic field. Thus, at the mean field level the system can be described as a Fermi liquid of composite fermions.

In this thesis, the effect of the gauge-field fluctuations around the mean-field Fermi-liquid state has been studied. It turns out that singular behavior appears in the lowest-order self-energy correction of fermions by the transverse gauge-field fluctuation. This singular-self energy correction makes the effective mass of the fermion divergent so that the usual single particle picture breaks down.

However, the one-particle Green's function is not gauge-invariant, so the singular self-energy could be an artifact of the gauge choice. This consideration leads us to examine the lowest order perturbative corrections to the gauge-invariant density-density and the current-current correlation functions. It is found that there are important cancellations between the self-energy corrections and the vertex corrections due to the Ward-identity. As a result, the density-density and the current-current correlation functions show a Fermi-liquid behavior for all ratios of ω and $v_F q$. From these results, one may suspect whether the divergent mass obtained from the self-energy has any physical meaning.

In order to answer the question about the effective mass, it is important to examine other gauge-invariant quantities which may potentially show a divergent effective mass. We have calculated the corrections to the activation energy gap and the corresponding effective mass by looking at the compressibility of the system. They are turned out to be singular and consistent with the previous self-consistent treatment of the self-energy. Therefore, the divergent effective mass does have a physical meaning.

At this point, it is clear that we need a unified framework to understand the apparently different behaviors of these two diffrent results. In order to achieve this, we construct a quantum Boltzman equation (QBE) which describes all the low energy physics of the composite fermion system, and provide consistent explanation for the previous calculations of response functions.

Thesis Supervisor: Xiao-Gang Wen
Title: Professor of Physics

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Chapter 1

Introduction

1.1 Background and Motivation

Since the discovery of the integer (IQH) and fractional quantum Hall (FQH) effects the two-dimensional electron system in strong magnetic fields has often surprised us. Among recent developements, a lot of attention has been given to the appearance of the new metallic state at the filling fraction $\nu = 1/2$ [1] and the associated Shubnikov-de Haas oscillations of the longitudinal resistance around $\nu = 1/2$ [2, 3]. The similarity between these phenamena near $\nu = 1/2$ and those of electrons in weak magnetic fields was successfully explained by the composite fermion approach [4]. Using the fermionic Chern-Simons gauge theory of the composite fermions [5, 6], Halperin, Lee, and Read (HLR) developed a theory that describes the new metallic state at $\nu = 1/2$ [6].

A composite fermion is obtained by attaching an even number $2n$ of flux quanta to an electron and the transformation can be realized by introducing an appropriate Chern-Simons gauge field [4, 5, 6]. At the mean field level, one takes into account only the average of the statistical magnetic field due to the attached magnetic flux. If the interaction between fermions is ignored, the system can be described as the free fermions in an effective magnetic field $\Delta B = B - B_{1/2n}$, where $B_{1/2n} = 2nn_e hc/e$ is the averaged statistical magnetic field and n_e is the density of electrons. Therefore, in the mean field theory, the FQH states with $\nu = \frac{p}{2np+1}$ can be described as the IQH state of the composite fermions with p filled Landau levels occupied in an effective magnetic field ΔB [4, 5, 6]. In particular, $\Delta B = 0$ at the filling fractions $\nu = 1/2n$ so that the ground state of the system is the filled Fermi sea with a well defined Fermi wave vector k_F [6, 7]. As a result, the Shubnikov-de Haas oscillations near $\nu = 1/2$ can be explained by the presence of a well defined Fermi wave vector at $\nu = 1/2$ [6]. The mean field energy gap of the system with $\nu = \frac{p}{2p+1}$ in the $p \to \infty$ limit is given by $E_g = \frac{e\Delta B}{mc}$, where m is the mass of the composite fermions. Note that, in the large ω_c limit, the finite m is caused by the Coulomb interaction between the fermions. The effective mass m should be chosen such that the Fermi energy E_F is given by the Coulomb energy scale.

There are a number of experiments which show that there is a well defined Fermi wave vector at $\nu = 1/2$ [8, 9, 10]. They observed the geometrical resonances between the semiclassical orbit of the composite fermions and another length scale artificially introduced to the system near $\nu = 1/2$.

However, it is possible that the fluctuations and the two-particle interactions, which are ignored in the mean field theory, are very important. Note that the density fluctuations correspond to the fluctuations of the statistical magnetic field. Therefore, the density fluctuations above the mean field state induces the gauge field fluctuations [5, 6]. If the fermions are interacting via a two-particle interaction $v(\mathbf{q}) = V_0/q^{2-\eta}(1 \leq \eta \leq 2)$, the effects of the gauge field fluctuations can be modified. In fact, the gauge field fluctuations become more singular as the interaction range becomes shorter (larger η). The reason is that the longer range interaction (smaller η) suppresses more effectively the density fluctuations, thus it induces the less singular gauge field fluctuations. Therefore, it is important to examine whether the mean field Fermi-liquid state is stable against the gauge field fluctuations which also includes the effects of the two-particle interaction.

One way to study the stability of the mean field Fermi-liquid state is to examine the low energy behavior of the self-energy correction induced by the gauge field fluctuations. It is found that the most singular contribution to the self-energy $\Sigma(\mathbf{k}, \omega)$ comes from the transverse part of the gauge field fluctuations [6, 11]. The lowest order perturbative correction to the self-energy (due to the transeverse gauge field) is calculated by several authors [6, 11, 13]. It turns out that Re $\Sigma \sim$ Im $\Sigma \sim \omega^{\frac{2}{1+\eta}}$ for $1 < \eta \leq 2$ and Re $\Sigma \sim \omega \ln \omega$, Im $\Sigma \sim \omega$ for $\eta = 1$ (Coulomb interaction). Thus the Landau criterion for the quasi-particle is violated in the case of $1 < \eta \leq 2$ and the case of $\eta = 1$ shows the marginal Fermi liquid behavior. In either cases, the effective mass of the fermions diverges, as $m^*/m \propto |\xi_k|^{-\frac{\eta-1}{\eta+1}}$ for $1 < \eta \leq 2$ and as $m^*/m \propto |\ln \xi_k|$ for $\eta = 1$, where $\xi_k = \frac{k^2}{2m} - \mu$ and μ is the chemical potential [6].

In a self-consistent treatment of the self-energy [6, 13], the energy gap of the system in the presence of a small effective magnetic field ΔB can be determined as $E_{\rm g} \propto |\Delta B|^{\frac{1+\eta}{2}}$ for $1 < \eta \leq 2$ and $E_{\rm g} \propto \frac{|\Delta B|}{|\ln \Delta B|}$ for $\eta = 1$. Therefore, the energy gap of the system vanishes faster than the mean field prediction or equivalently the effective mass diverges in a singular way as $\nu = 1/2$ is approached. These results suggest that the effective Fermi velocity of the fermion v_F^* goes to zero at $\nu = 1/2$ even though the Fermi wave vector k_F is finite and the quasi-particles have a very short life time $\tau \approx (T/\varepsilon_F)^{-2/(1+\eta)}(1/\varepsilon_F)$, where T is the temperature and ε_F is the Fermi energy. However, the recent magnetic focusing experiment [10] suggests that the fermion has a long life time or a long mean free path which seems inconsistent with the above picture.

In particular, since the one-particle Green's function is not gauge-invariant, the singular self-energy could be an artifact of the gauge choice. Therefore, it is very important to study gauge-invariant objects like two-particle Green's functions or response functions to see what are the conditions under which the divergent effective mass is observable. One important question is whether the divergent effective mass obtained from the self-energy correction has any physical meaning. If it has a physical significance, it is also necessary to reconcile the result of the self-energy calculation and the existing experiments. It is the purpose of this thesis to propose a unified framework to understand these theoretical and experimantal findings. In the next section, we introduce the model and review one-particle Green's function and the lowest order perturbative correction to the self-energy. In section 1.3, we will briefly summarize the important results we got and give the outline of the thesis.

1.2 The Model and the One-Particle Properties

The two dimensional electrons interacting via a two-particle interaction can be transformed to the composite fermions interacting via the same two-partice interaction and also coupled to an appropriate Chern-Simons gauge field which appears due to the statistical magnetic flux quanta attached to each electron [5, 6]. The model can be constructed as follows ($\hbar = e = c = 1$).

$$Z = \int D\psi \; D\psi^* \; Da_\mu \; e^{\, i \int dt \; d^2r \; \mathcal{L}} \; , \tag{1.1}$$

where the Lagrangian density $\mathcal{L}$ is

$$\begin{aligned} \mathcal{L} &= \psi^*(\partial_0 + ia_0 - \mu)\psi - \frac{1}{2m}\psi^*(\partial_i - ia_i + iA_i)^2\psi \\ &\quad - \frac{i}{2\pi\tilde{\phi}} \; a_0 \varepsilon^{ij}\partial_i a_j + \frac{1}{2}\int d^2r' \; \psi^*(\mathbf{r})\psi(\mathbf{r}) \; v(\mathbf{r}-\mathbf{r}') \; \psi^*(\mathbf{r}')\psi(\mathbf{r}') \; , \end{aligned} \tag{1.2}$$

where ψ represents the fermion field and $\tilde{\phi}$ is an even number $2n$ which is the number of flux quanta attached to an electron, and $v(\mathbf{r}) \propto V_0/r^\eta$ is the Fourier transform of $v(\mathbf{q}) = V_0/q^{2-\eta}$ ($1 \le \eta \le 2$) which represents the interaction between the fermions. $\mathbf{A}$ is the external vector potential ($B = \nabla \times \mathbf{A}$) and we choose the Coulomb gauge $\nabla \cdot \mathbf{a} = 0$ for the Chern-Simons gauge field. Note that the integration over a_0 enforces the following constraint:

$$\nabla \times \mathbf{a} = 2\pi\tilde{\phi} \; \psi^*(\mathbf{r})\psi(\mathbf{r}) \; , \tag{1.3}$$

which represents the fact that $\tilde{\phi}$ number of flux quanta are attached to each electron.

The saddle point of the action is given by the following conditions:

$$\nabla \times \langle \mathbf{a} \rangle = 2\pi\tilde{\phi} \; n_e = B_{1/2n} \quad \text{and} \quad \langle a_0 \rangle = 0 \; . \tag{1.4}$$

Therefore, at the mean field level, the fermions see an effective magnetic field ($\Delta\mathbf{A} = \mathbf{A} - \langle \mathbf{a} \rangle$):

$$\Delta B = \nabla \times \Delta\mathbf{A} = B - B_{1/2n} \; , \tag{1.5}$$

which becomes zero at the Landau level filling factor $\nu = 1/2n$. The IQH effect of the fermions may appear when the effective Landau level filling factor $p = \frac{2\pi n_e}{\Delta B}$ becomes an integer. This implies that the real external magnetic field is given by $B = B_{1/2n} + \Delta B = 2\pi n_e \left(\frac{2np+1}{p}\right)$ which corresponds to a FQH state of electrons with the filling factor $\nu = \frac{p}{2np+1}$.

The fluctuations of the Chern-Simons gauge field, $\delta a_\mu = a_\mu - \langle a_\mu \rangle$, can be incorperated as follows.

$$Z = \int D\psi\ D\psi^*\ D\delta a_\mu\ e^{\,i \int dt\ d^2r\ \mathcal{L}} \ , \tag{1.6}$$

where

$$\begin{aligned}\mathcal{L} &= \psi^*(\partial_0 + i\ \delta a_0 - \mu)\psi - \frac{1}{2m}\psi^*(\partial_i - i\ \delta a_i + i\ \Delta A_i)^2\psi - \frac{i}{2\pi\tilde{\phi}}\ \delta a_0\ \varepsilon^{ij}\ \partial_i\ \delta a_j \\ &+ \frac{1}{2(2\pi\tilde{\phi})^2}\int d^2r'\ (\nabla\times\delta\mathbf{a}(\mathbf{r}))\ v(\mathbf{r}-\mathbf{r}')\ (\nabla\times\delta\mathbf{a}(\mathbf{r}'))\ ,\end{aligned} \tag{1.7}$$

After integrating out the fermions and including gauge field fluctuations within the random phase approximation (RPA) [6], the effective action of the gauge field can be obtained as

$$S_{\text{eff}} = \frac{1}{2}\int \frac{d^2q}{(2\pi)^2}\ \frac{d\omega}{2\pi}\ \delta a^*_\mu(\mathbf{q},\omega)\ D^{-1}_{\mu\nu}(\mathbf{q},\omega,\Delta B)\ \delta a_\nu(\mathbf{q},\omega)\ , \tag{1.8}$$

where $D^{-1}_{\mu\nu}(\mathbf{q},\omega,\Delta B)$ was calculated by several authors [6, 36, 37]. For our purpose, the 2×2 matrix form for $D^{-1}_{\mu\nu}$ is sufficient so that $\mu,\nu = 0,1$ and 1 represents the direction that is perpendicular to $\mathbf{q}$. In particular, when $\Delta B = 0$, the gauge field propagator has the following form [6]:

$$D^{-1}_{\mu\nu}(\mathbf{q},\omega) = \begin{pmatrix} \frac{m}{2\pi} & -i\frac{q}{2\pi\tilde{\phi}} \\ i\frac{q}{2\pi\tilde{\phi}} & -i\gamma\frac{\omega}{q} + \tilde{\chi}(q)q^2 \end{pmatrix}\ , \tag{1.9}$$

where $\gamma = \frac{2n_e}{k_F}$ and $\tilde{\chi}(q) = \frac{1}{24\pi m} + \frac{v(q)}{(2\pi\tilde{\phi})^2}$. Since the most singular contribution to the self-energy correction comes from the transverse part of the gauge field [6, 11], we concentrate on the effect of the transverse gauge field fluctuations. In the infrared limit, the transeverse gauge field propagator can be taken as [6, 13, 15]

$$D_{11}(\mathbf{q},\omega) = \frac{1}{-i\gamma\frac{\omega}{q} + \chi q^\eta}\ , \tag{1.10}$$

where $\chi = \frac{1}{24\pi m} + \frac{V_0}{(2\pi\tilde{\phi})^2}$ for $\eta = 2$ and $\chi = \frac{V_0}{(2\pi\tilde{\phi})^2}$ for $1 \le \eta < 2$.

Since the calculations in chapter 2 are done in Euclidean space-time, it is worthwhile to explain the derivation of the gauge field propagator in Euclidean formalism. The effective action of the gauge field is now given by [6, 11, 12]

$$S_{\text{eff}} = \frac{1}{2}\int \frac{d^2q}{(2\pi)^2}\frac{d\omega}{2\pi}\ \delta a^*_\mu(\mathbf{q},\omega)\ D^{-1}_{\mu\nu}(\mathbf{q},i\omega)\ \delta a_\nu(\mathbf{q},\omega)\ , \tag{1.11}$$

where

$$D^{-1}_{\mu\nu} = \begin{pmatrix} \Pi^0_{00} & -\frac{q}{2\pi\tilde{\phi}} \\ \frac{q}{2\pi\tilde{\phi}} & \Pi^0_{11} + \frac{1}{(2\pi\tilde{\phi})^2}v(q)q^2 \end{pmatrix}\ . \tag{1.12}$$

Π^0_{00} and Π^0_{11} are given by the one-loop diagrams in Figures 1-1 (a) and (b) respectively.

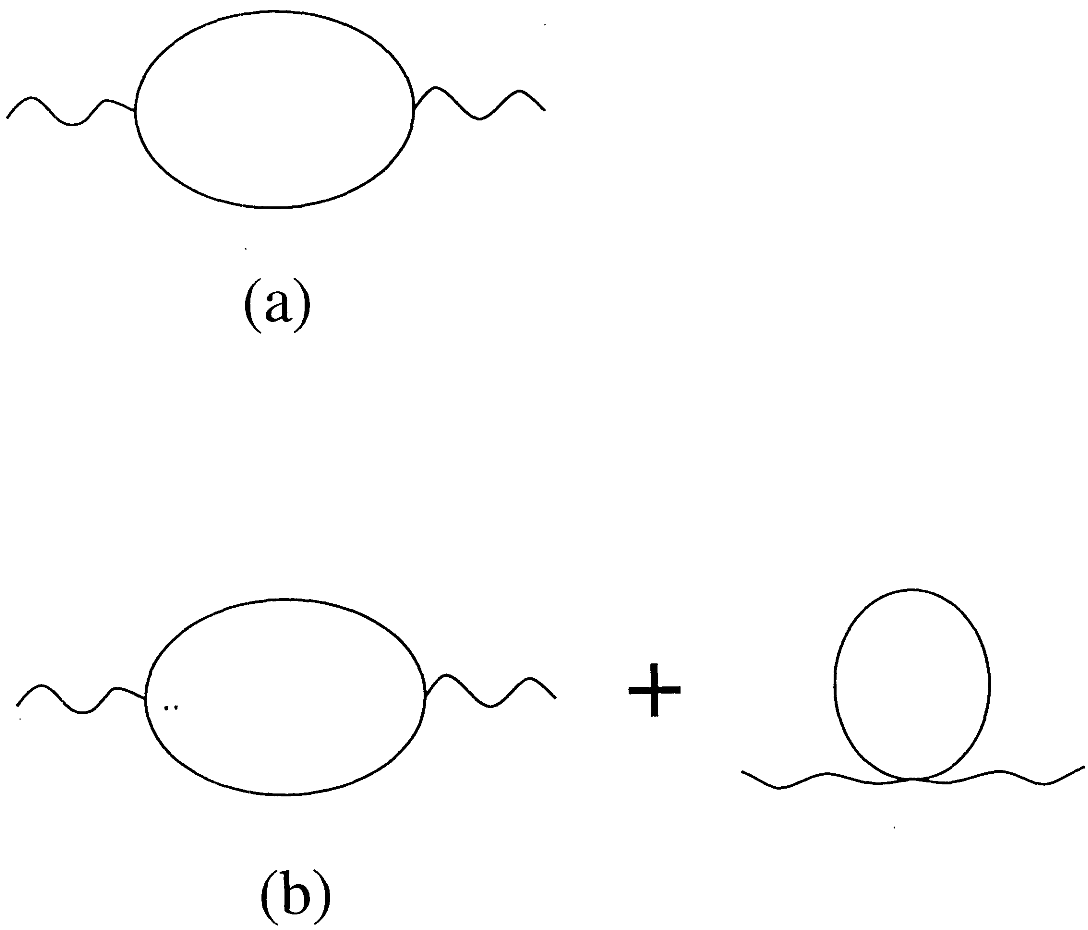


Figure 1-1: The one-loop diagrams for Π^0_{00} (a) and for Π^0_{11} (b). The solid line is the bare fermion propagator and the wavy line represents the gauge field propagator. These are the leading order diagrams of Π_{00} and Π_{11} in the $1/N$ expansion of a large N generalized theory.

In the limit of $\omega \ll v_F q$, one can find that [6, 11, 12]

$$
\begin{aligned}
\Pi^0_{00} &= -\frac{m}{2\pi}\left(1 - \frac{|\omega|}{v_F q}\right) \\
\Pi^0_{11} &= \frac{2n}{k_F}\frac{|\omega|}{q} + \frac{q^2}{24\pi m} \\
&\equiv \gamma\frac{|\omega|}{q} + \chi_0 q^2 \ .
\end{aligned}
\tag{1.13}
$$

Therefore, the diagonal components of the gauge field propagator can be expressed as

$$
\begin{aligned}
D^{-1}_{00} &= -\frac{m}{2\pi}\left(1 - \frac{|\omega|}{v_F q}\right) \\
D^{-1}_{11} &= \gamma\frac{|\omega|}{q} + \tilde{\chi}(q) q^2 \\
&\approx \gamma\frac{|\omega|}{q} + \chi q^\eta \ ,
\end{aligned}
\tag{1.14}
$$

where χ is the same as that in Eq. (1.10).

Since the longitudinal part of the gauge field is screened, the transverse part of the gauge field dominates the physics. The one-loop self energy correction (in Euclidean space) due to the transverse part of the gauge field is calculated as (Figure 1-2) [6, 11, 13]

$$
\begin{aligned}
\Sigma(\mathbf{k}, i\omega) &= \int \frac{d^2 q}{(2\pi)^2}\frac{d\nu}{2\pi}\left|\frac{\mathbf{k}\times\hat{\mathbf{q}}}{m}\right|^2 G_0(\mathbf{k}+\mathbf{q}, i\omega + i\nu)\ D_{11}(\mathbf{q}, i\nu) \\
&\approx -i\ \lambda\ |\omega|^{\frac{2}{1+\eta}}\ \mathrm{sgn}(\omega)\ ,
\end{aligned}
\tag{1.15}
$$

where

$$
\lambda = \frac{v_F}{4\pi\ \sin(\frac{2\pi}{1+\eta})\ \gamma^{\frac{\eta-1}{\eta+1}}\ \chi^{\frac{2}{1+\eta}}}\ , \tag{1.16}
$$

and $G_0^{-1}(\mathbf{k}, i\omega) = i\omega - \xi_{\mathbf{k}}$ ($\xi_{\mathbf{k}} = \frac{k^2}{2m} - \mu$). The self energy as a function of real frequency Ω (in the Minkowski space) can be obtained from the analytic continuation of $\Sigma(\mathbf{k}, i\omega)$, *i.e.*, $\Sigma(\mathbf{k}, \Omega) = \Sigma(\mathbf{k}, i\omega \to \Omega + i\delta)$. Note that $|\mathrm{Im}\ \Sigma(\mathbf{k}, \Omega)| \propto |\Omega|^{\frac{2}{1+\eta}} \gg |\Omega|$ for sufficiently small Ω or $|\Omega| \ll \lambda^{\frac{\eta+1}{\eta-1}}$ ($\eta > 1$). Therefore, the quasi-particle (the dressed fermion) is not well defined.

This can be also seen from the spectral function of fermions. The spectral function can be obtained from the imaginary part of the retarded Green's function: $A(\mathbf{k}, \Omega) = -\frac{1}{\pi}\ \mathrm{Im}\ G_R(\mathbf{k}, \Omega) = -\frac{1}{\pi}\ \mathrm{Im}\ G(\mathbf{k}, i\omega \to \Omega + i\delta)$, where $G^{-1}(\mathbf{k}, i\omega) = G_0^{-1}(\mathbf{k}, i\omega) - \Sigma(\mathbf{k}, i\omega)$. In the low frequency limit,

$$
A(\mathbf{k}, \Omega) \approx \frac{1}{\pi}\ \frac{\lambda_2\ |\Omega|^{\frac{2}{1+\eta}}\ \mathrm{sgn}(\Omega)}{(\lambda_1\ |\Omega|^{\frac{2}{1+\eta}} - \xi_{\mathbf{k}})^2 + (\lambda_2\ |\Omega|^{\frac{2}{1+\eta}})^2}\ , \tag{1.17}
$$

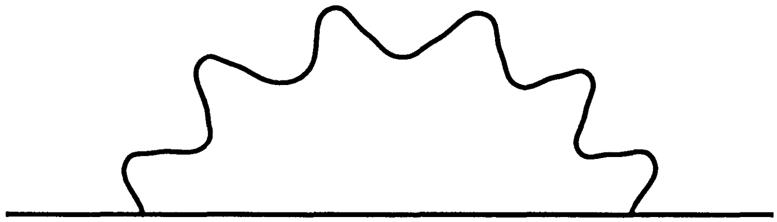


Figure 1-2: The diagram that corresponds to the one-loop correction to the fermion self-energy. The solid line is the bare fermion propagator and the wavy line represents the gauge field propagator

where $\lambda_1 = \lambda \cos\left[\frac{\pi}{2}\left(\frac{\eta-1}{\eta+1}\right)\right]$ and $\lambda_2 = \lambda \sin\left[\frac{\pi}{2}\left(\frac{\eta-1}{\eta+1}\right)\right]$ for $\eta > 1$. Note that the maximum of $A(\mathbf{k}, \Omega)$ appears at $\Omega \sim \left(\frac{\xi_{\mathbf{k}}}{\lambda_1}\right)^{\frac{1+\eta}{2}}$. However, the width of the broad peak is also order $\Delta\Omega \sim \left(\frac{\xi_{\mathbf{k}}}{\lambda_1}\right)^{\frac{1+\eta}{2}}$. Therefore, the Landau criterion for the existence of quasi-particles ($\Delta\Omega \ll \Omega$) is marginally violated.

If we assumed that there is a well-defined Fermi wave vector $k_F = (4\pi n_f)^{1/2}$ and tried to fit the result to the usual quasi-particle picture, the energy spectrum of the quasi-particle would be [6, 13]

$$\epsilon_{\mathbf{k}} \propto |k - k_F|^{\frac{1+\eta}{2}} \tag{1.18}$$

for k sufficiently close to k_F. From $\frac{k_F}{m^*} = \left.\frac{\partial \epsilon_{\mathbf{k}}}{\partial k}\right|_{k=k_F}$, the effective mass diverges as

$$m^* \propto |k - k_F|^{-\frac{\eta-1}{2}} \propto |\epsilon_{\mathbf{k}}|^{-\frac{\eta-1}{\eta+1}} . \tag{1.19}$$

This suggests that at least some modifications to the conventional Fermi-liquid theory are necessary as far as the one-particle Green's function is concerned.

There have been also some nonperturbative calculations of the one-particle Green's function [17, 18, 19, 23], which were motivated by the singular perturbative correction at low energies. The results look very different from that obtained by the lowest order perturbative calculation and even exponentially decaying one-particle Green's function is found in the so-called eikonal limit [19].

From these results, one may doubt the validity of the quasi-particle picture although a modified Fermi liquid description is proposed [6]. However, one should also remember that the one-particle Green's function is not gauge invariant. This can be easily seen in the path integral representation of the one-particle Green's function [11, 17] of a fermion interacting with a gauge field, *i.e.*, each path acquires a phase factor $e^{i\int_0^t dt' \mathbf{a}(\mathbf{r},t')\cdot d\mathbf{r}/dt'}$ which is manifestly not gauge invariant. Therefore, it is very important to examine gauge-invariant quantities. One of the purposes of this thesis is examine these gauge-invariant quantities.

1.3 Overview of the Results and Outline of the Thesis

As mentioned in section 1.1, the one-particle Green's function is not gauge-invariant, so the singular self-energy could be an artifact of the gauge choice. In chapter 2, we address this question and examine the lowest order perturbative corrections to the gauge-invariant density-density and the current-current correlation functions [13]. It is found that there are important cancellations between the self-energy corrections and the vertex corrections due to the Ward-identity [13, 14]. As a result, the density-density and the current-current correlation functions show a Fermi-liquid behavior for all ratios of ω and $v_F q$ [13]. In particular, the edge of the particle-hole continuum $\omega = v_F q$ is essentially not changed, which may suggest a finite effective mass. From the current-current correlation function, the transport scattering rate (due to the transverse part of the gauge field) is given by $1/\tau_{\rm tr} \propto \omega^{\frac{4}{1+\eta}} \ll \omega$ after the cancellation (The scattering rate would be much larger $1/\tau_{\rm tr} \propto \omega^{\frac{2}{1+\eta}} \gg \omega$ had we ignored the vertex correction) [13]. Therefore, the fermions have a long transport life time which explains a long free path in the magnetic focusing experiment. From these results, one may suspect whether the divergent mass obtained from the self-energy has any physical meaning.

However, due to the absence of the underlying quasi-particle picture, we cannot simply conclude that the fermions have a finite effective mass associated with the long life time which was obtained from the small q and ω behaviors of the density-density and the current-current correlation functions. In fact, it is found that $2k_F$ response functions show singular behaviors compared to the usual Fermi liquid theory [14]. We also like to mention that the recent experiments on the Shbunikov-de Hass oscillations [3] have observed some features which were interpretated as a sign of the divergent effective mass of the fermions as $\nu = 1/2$ is approached. The experimentally determined effective mass diverges in a more singular way than any theoretical prediction. However, their determination of the effective mass is based on a theory for the non-interacting fermions and also the disorder effect is very important near $\nu = 1/2$ because the static fluctuations of the density due to the impurities induces an additional static random magnetic field. Since there is no satisfactory theory in the presence of disorder, it is difficult to compare the present theory and the experiments.

In order to answer the question about the effective mass, it is important to examine other gauge-invariant quantities which may potentially show a divergent effective mass. In chapter 3, we calculated the lowest order perturbative correction to the compressibility with a fixed ΔB, which shows a thermally activated behavior when the chemical potential lies exactly at the middle of the successive effective Landau levels [15]. It turns out that the corrections to the activation energy gap and the corresponding effective mass are singular and consistent with the previous self-consistent treatment of the self-energy [6]. Thus it is necessary to understand the apparently different behaviors of the density-density correlation function at $\nu = 1/2$ and the activation energy gap determined from the compressibility near $\nu = 1/2$.

One resolution of the problem was suggested by Stern and Halperin [16] within the usual Landau-Fermi-liquid theory framework. The idea is that both of the effective mass and the Landau-interaction-function are singular in such a way that they cancel each other in the density-density correlation function. Recently, Stern and Halperin [16] put forward this idea and construct a Fermi-liquid-theory of the fermion-gauge system in the case of Coulomb interaction. Even though the use of the Landau-Fermi-liquid theory or equivalently the existence of the well defined quasi-particles can be *marginally* justified in the case of the Coulomb interacion, it is necessary to construct a more general framework which applies to the arbitrary two-particle interaction ($1 < \eta \leq 2$ as well as $\eta = 1$) and allows us check the validity of the Fermi liquid theory and to judge when the divergent mass shows up. In particular, it is worthwhile to provide a unified picture for understanding the previous theoretical studies.

In the usual Fermi-liquid theory, the quantum Boltzmann equation (QBE) of the quasi-particles provides the useful informations about the low lying excitations of the system. In chapter 4, we construct a similar QBE which describes all the low energy physics of the composite fermion system. We use the non-equilibrium Green's function technique [32, 33, 34, 35] to derive the new QBE and calculate the generalized Landau-interaction-function which has the frequency dependence as well as the usual angular dependence due to the retarded nature of the gauge interaction.

By studing the dynamical properties of the collective modes using the QBE, we find that the smooth fluctuations of the Fermi surface show the usual Fermi-liquid behavior, while the rough fluctuations show the singular behavior determined by the singular self-energy correction. From these results, we find that the density-density and the current-current correlation functions, being dominated by the smooth fluctuations of the Fermi surface, show the usual Fermi-liquid behavior. On the other hand, the energy gap away from $\nu = 1/2$ is determined by the behaviors of the rough fluctuations of the Fermi surface so that the singular mass correction shows up in the energy gap of the system.

The outline of the thesis is as follows. In chapter 2, gauge-invariant density-density and current-current correlation functions (at $\nu = 1/2$) are examined in the long wavelength and the low frequency limit. In chapter 3, we calculate the finite temperature compressibility of the system near $\nu = 1/2$ and extract the information about the energy gap of the system. In chapter 4, the QBE of the fermion-gauge-field system is derived and is used to explain the previous theoretical and experimental findings.

Chapter 2

Gauge-invariant Response Functions at $\nu = 1/2$

2.1 Introduction

The problem of two-dimensional fermions interacting with a gauge field appears also in a gauge theory of the normal state of high temperature superconductors (HTSC) [11, 12], which corresponds to the case of $\eta = 2$. Besides the real examples, this problem has been studied as a potential example of non-Fermi liquids [21, 17, 18, 19, 23, 25, 24, 22, 20]. This is due to the fact that, as shown in chapter 1, the transverse part of the gauge field makes the self-energy correction singular. Note that, even in the lowest order, the singular self energy correction makes the effective mass of the fermion divergent so that the usual single particle picture breaks down [6].

In this chapter, we concentrate on the effects of the transverse part of the gauge field to the gauge-invariant correlation functions at $\nu = 1/2$. In contrast to the usual long-range interactions such as the Coulomb interaction, the transverse part of the gauge field cannot be screened because the gauge invariance requires the gauge field to be massless in the absence of symmetry breaking [21, 27, 28]. Thus, one can expect that the long-range interaction due to the transverse part of the gauge field may give rise to non-Fermi-liquid-like behaviors. Note that the singular self-energy correction makes perturbative calculation unreliable at low energies. This motivated several non-perturbative calculations of one-particle Green's function of fermions which show highly non-Fermi-liquid-like behaviors [17, 18, 19, 23].

However, the singular self-energy correction in the one-particle Green's function (which leads to divergent effective mass [6]) could be an artifact of the gauge choice rather than a property of physical quasi-particles. It is possible that some singularities in the gauge-dependent one-particle Green's function simply do not appear in gauge-invariant correlation functions. One purpose of this chapter is to examine some gauge-invariant response functions in order to determine whether the singular behaviors in the one-particle Green's function appear in gauge-invariant correlation functions or not.

The importance of the gauge-invariance in calculating correlation functions can be also seen in the following example. The leading order corrections (up to two-loop level) to the transverse polarization function (or current-current correlation function) are given by the diagrams in Figure 2-1 (a)-(d).

Note that the sum of contributions from Figures 2-1 (a)-(d) is not gauge-invariant because they contain only self-energy corrections but do not contain the vertex correction. For concreteness, let us consider the case of $\eta = 2$, which corresponds to the case of HTSC and the short-range interaction between fermions in HFLL. We also consider $\Omega \ll v_F q$ and $q \ll k_F$ limits. In this case, it can be shown that the correction to the transverse polarization function due to the self-energy corrections (given by Fig. 2-1 (a)-(d)) has the following form:

$$\delta \text{ Im } \Pi^s_{11}(\mathbf{q}, \Omega) \approx \frac{m^2 v_F^3}{2\pi\gamma} \frac{\Omega}{v_F q} \frac{(\gamma\Omega/\chi)^{2/3}}{k_F q} , \tag{2.1}$$

while the contribution from the free fermions is given by

$$\text{Im } \Pi^0_{11}(\mathbf{q}, \Omega) = -\frac{m v_F^2}{2\pi} \frac{\Omega}{v_F q} , \tag{2.2}$$

where 1 denotes the direction which is perpendicular to $\mathbf{q}$. One can see that the correction δ Im Π^s_{11} would be more singular than the free fermion contribution Im Π^0_{11} if q, $\Omega \to 0$ limit was taken with fixed $\Omega/v_F q < 1$. This result suggests that the perturbative expansion breaks down at low energies and the Fermi-liquid criterion are violated. Thus the gauge-dependent correction (which comes from the self-energy correction) to the transverse polarization function provides a similar picture as that from the singular one-particle Green's function [29].

Nevertheless, the perturbative corrections to the correlation functions should be calculated in a gauge-invariant way, thus one has to include the contributions from the vertex correction. The contribution to the transverse polarization function δ Im Π^v_{11} coming from the vertex correction contains a singular term which exactly cancels the singular contribution from the self-energy correction. Thus, the remnant terms in δ Im Π^v_{11} provide the lowest order corrections to the transverse polarization function and have the following form:

$$\delta \text{ Im } \Pi^s_{11} + \delta \text{ Im } \Pi^v_{11} \approx \frac{m^2 v_F^3}{\gamma} \frac{\Omega}{v_F q} \left[a \, \frac{(\gamma\Omega/\chi)^{2/3}}{k_F^2} + b \, \frac{(\gamma\Omega/\chi)}{k_F^2 q} \right] , \tag{2.3}$$

where a, b are dimensionless constants. One can see that the corrections calculated in a gauge-invariant way are always much less than the free fermion contribution as far as $\Omega \ll v_F q$ and $q \ll k_F$ limits are concerned. Therefore, the perturbative expansion works quite well in this regime, at least up to the leading order gauge field corrections, and there is no need to go beyond the perturbation theory at this order. The validity of the perturbative expansion also indicates that the transverse polarization function is well described by the Fermi-liquid theory in the region of $\Omega \ll v_F q$ and $q \ll k_F$. This provides a very different picture from that obtained through the gauge-dependent one-particle Green's function.

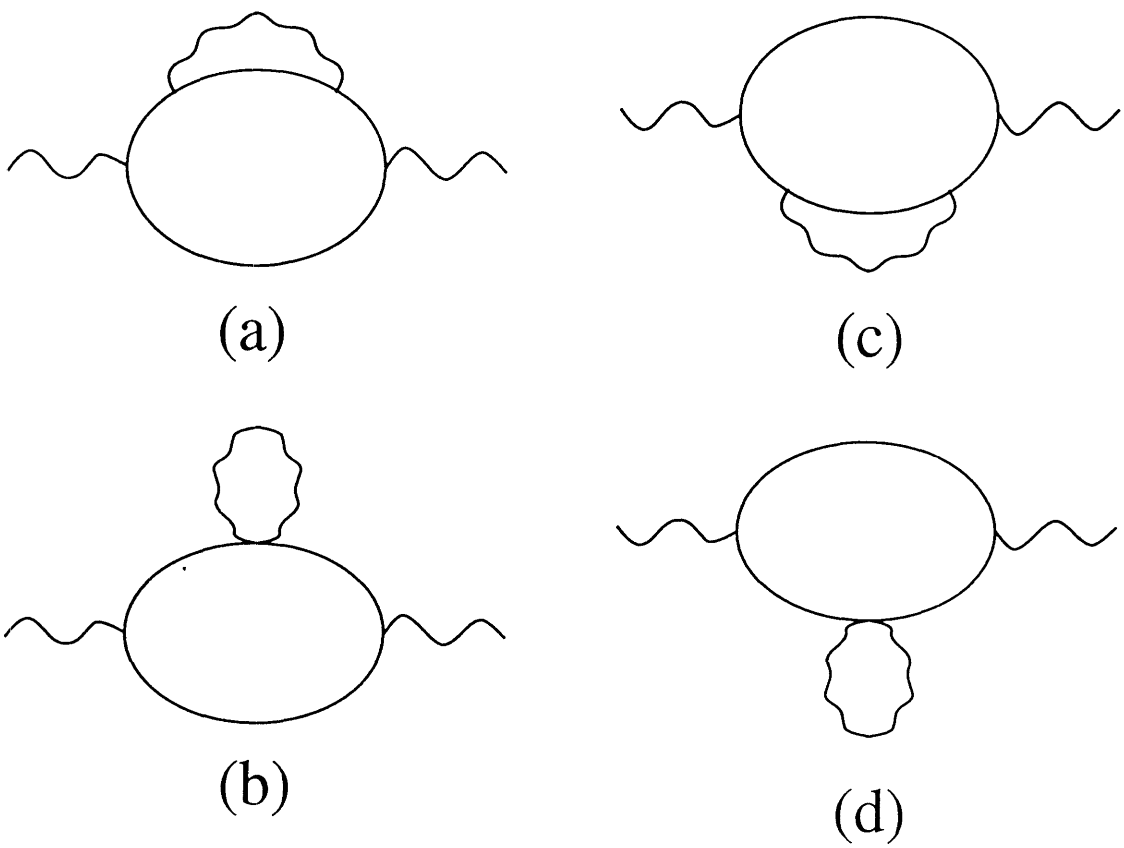
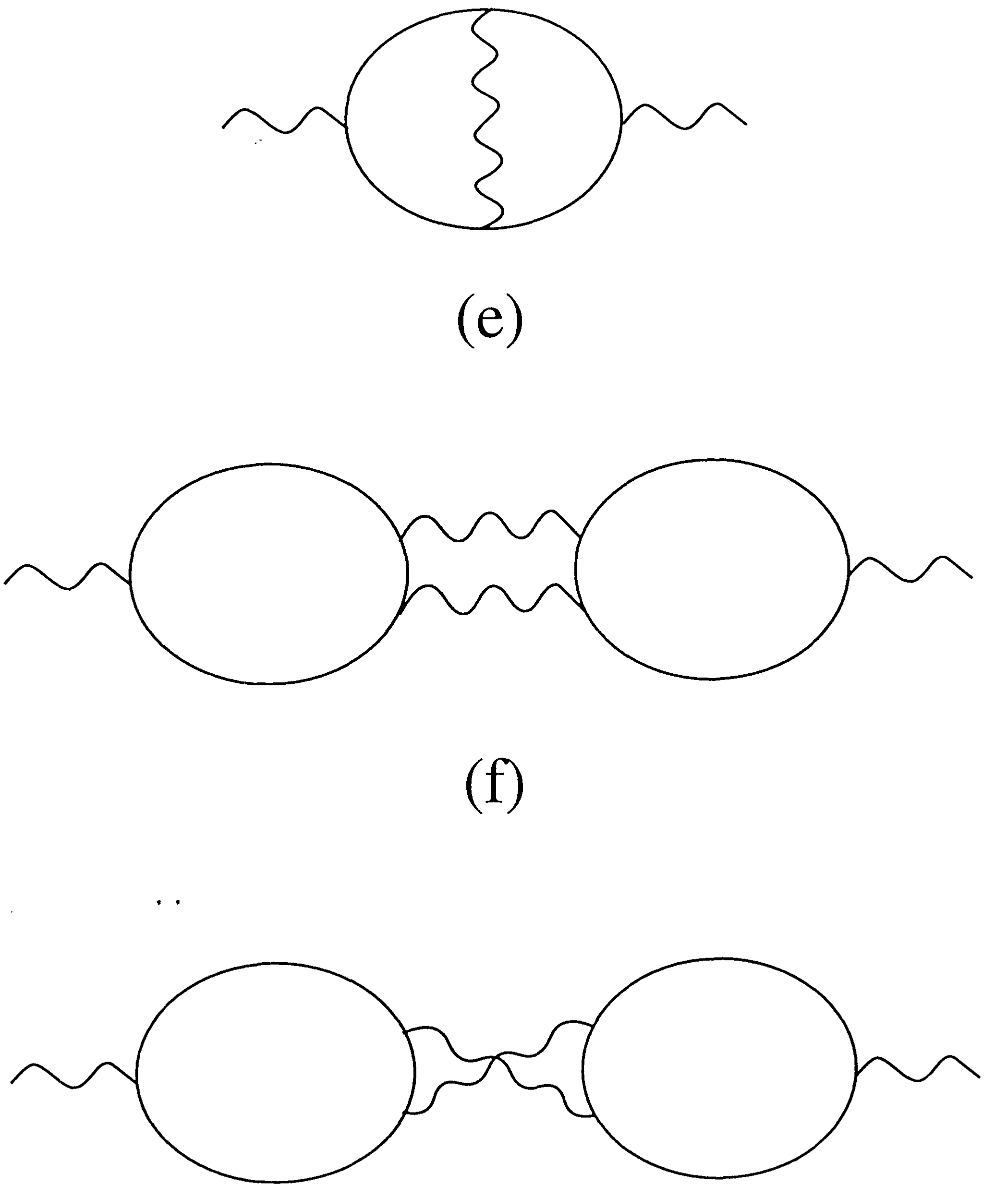


Figure 2-1: The diagrams that correspond to the $(1/N)^0$th order contributions to Π_{11} in the $1/N$ expansion. In the coupling constant expansion, (a)-(d) correspond to the two-loop diagrams.



(e)

(f)

(g)

Figure 2-1: The diagrams that correspond to the $(1/N)^0$th order contributions to Π_{11} in the $1/N$ expansion (continued).

In this chapter, we examine several gauge-invariant two-particle Green's functions or response functions in the limit of low frequency and long wavelength. It is shown that all the leading singular contributions from the self-energy correction are cancelled by the contributions from the vertex correction in systematic perturbation theories (which guarantee the gauge-invariance in each order of the perturbative expansion). This cancellation is essentially due to the Ward identity. It is found that singular corrections to the two-particle Green's function do not appear for all ratios of $\Omega/v_F q$ as far as the limit of low frequency and long wavelength limit is concerned. This kind of cancellation was also discussed by Ioffe and Kalmeyer [30] for a static gauge field. Recently, Khveshchenko and Stamp [19] performed non-perturbative calculations of one-particle and two-particle Green's functions using the so-called eikonal approximation. Even though they obtained a highly singular one-particle Green's function, the singularity does not show up in two-particle Green's functions for small q and Ω in this approximation.

We also show explicitly that the gauge field propagator is not renormalized by the fluctuations beyond RPA up to two-loop order. Non-renormalization of the gauge field propagator was first discussed by Polchinski [20] in the framework of a self-consistent approach. In this approach, it is assumed that the dispersion relation of fermions is given by $\omega \propto \xi_{\mathbf{k}}^{3/2}$ ($\xi_{\mathbf{k}} = k^2/2m - \mu$) and that of the gauge field is given by $\Omega \propto iq^3$, which are the results of one-loop corrections. Ignoring vertex correction by assuming the existence of a Migdal-type theorem, he showed that the assumed one-particle Green's function is self-consistent, and the polarization function is given by the same form as that of free fermions Im $\Pi^0_{11} = -(mv_F^2/2\pi)\ (\Omega/v_F q)$ for $\Omega < \gamma^{1/3}\chi^{2/3}q^{3/2}$. As a result, the gauge field propagator is not renormalized because the dispersion relation of the gauge field is given by $\Omega \propto iq^3$. However, we would like to remark that his result is quite different from those obtained in this paper. One can check that the polarization function in the self-consistent approach has a different from compared to that of Fermi liquid for $\Omega > \gamma^{1/3}\chi^{2/3}q^{3/2}$. However, in our perturbative calculation, the cancellation of anomalous terms from self-energy and vertex corrections leads to the result that the polarization functions have Fermi liquid forms for all q and Ω as far as both are small.

We have made several explicit calculations of two-particle Green's functions. In particular, we consider $v(\mathbf{q}) = V_0/q^{2-\eta}$ ($v(\mathbf{r}) \propto V_0/r^\eta$, $1 < \eta \leq 2$) which corresponds to the interaction between fermions in the problem of HFLL. We will present the non-analytic contributions (due to the gauge field fluctuations) to the two-particle Green's functions. The transverse polarization function $\Pi_{11}(\mathbf{q},\Omega)$ up to two-loop order is found to have the following form. For $\Omega \ll v_F q$, we get

$$\text{Im}\ \Pi_{11}(\mathbf{q},\Omega) \approx -\frac{mv_F^2}{2\pi}\,\frac{\Omega}{v_F q}\left[1 - a\,\frac{mv_F}{\gamma}\frac{(\gamma\Omega/\chi)^{\frac{2}{1+\eta}}}{k_F^2} - b\,\frac{mv_F}{\gamma}\frac{(\gamma\Omega/\chi)^{\frac{3}{1+\eta}}}{k_F^2 q}\right], \quad (2.4)$$

while for $\Omega \gg v_F q$,

$$\text{Im } \Pi_{11}(\mathbf{q},\Omega) \approx -\frac{1+\eta}{8\pi^2(5+\eta)} \frac{1}{\sin\left(\frac{2\pi}{1+\eta}\right)} \frac{v_F}{m} \frac{\gamma^{\frac{3-\eta}{1+\eta}}}{\chi^{\frac{4}{1+\eta}}} \Omega^{\frac{3-\eta}{1+\eta}} \left[1 + c\, m v_F^3 \left(\frac{\chi}{\gamma}\right)^{\frac{1}{1+\eta}} \frac{q^2}{\Omega^{\frac{2\eta+3}{\eta+1}}} \right], \tag{2.5}$$

where a, b, c are positive dimensionless constants.

The density-density correlation function $\Pi_{00}(\mathbf{q},\Omega)$ is also calculated. We have a formula valid for any ratio of $\Omega/v_F q$ as long as Ω and q are small (see Eq. (2.63)), but here we just discuss limiting cases. For $\Omega \ll v_F q$, we have

$$\text{Im } \Pi_{00}(\mathbf{q},\Omega) \approx -\frac{m}{2\pi} \frac{\Omega}{v_F q} \left[1 - \frac{1+\eta}{4\pi(5+\eta)} \frac{1}{\cos\left(\frac{\eta-1}{\eta+1}\pi\right)} \frac{1}{k_F m} \frac{\gamma^{\frac{3-\eta}{1+\eta}}}{\chi^{\frac{4}{1+\eta}}} \Omega^{\frac{3-\eta}{1+\eta}} \left(\frac{\Omega}{v_F q}\right)^2 \right]. \tag{2.6}$$

On the other hand, for $\Omega \gg v_F q$,

$$\text{Im } \Pi_{00}(\mathbf{q},\Omega) \approx -\frac{1+\eta}{8\pi^2(5+\eta)} \frac{1}{\sin\left(\frac{2\pi}{1+\eta}\right)} \frac{1}{k_F} \frac{\gamma^{\frac{3-\eta}{1+\eta}}}{\chi^{\frac{4}{1+\eta}}} \Omega^{\frac{3-\eta}{1+\eta}} \left(\frac{v_F q}{\Omega}\right)^2 . \tag{2.7}$$

Note that $\text{Im } \Pi_{11}(q \to 0,\Omega) = \frac{\Omega^2}{v_F^2 q^2} \text{Im } \Pi_{00}(q \to 0, \Omega)$ is satisfied as it should be. Eqs. (2.4)- (2.7) are the main results of this paper.

From the above gauge-invariant correlation functions, one can see that

1) The corrections are irrelevant in the small q and Ω limit regardless of the way how q and Ω approach to zero (for example, $q \to 0$ limit may be taken first or $\Omega \to 0$ first, etc.). Therefore, *non-perturbative calculations are not necessary.* However, the sub-leading contributions are in general non-analytic due to the long range nature of the gauge interaction. The non-analytic sub-leading terms may have some experimental consequences. For example, the NMR relaxation rate $1/T_1$ in the problem of HTSC can be determined from $\Pi_{00}(\mathbf{q},\Omega)$. At low temperatures we have

$$\frac{1}{T_1 T} \propto \lim_{\Omega \to T} -\frac{1}{\Omega} \sum_{\mathbf{q}} \text{Im } \Pi_{00}(\mathbf{q},\Omega) , \tag{2.8}$$

where Π_{00} plays the role of spin susceptibility in HTSC. Eq. (2.6) implies the following non-analytic correction to the free fermion result (only contributions from small $\mathbf{q}$ are considered) $\frac{1}{T_1 T} \propto 1 - A\, T^{\frac{5+\eta}{1+\eta}}$, where A is a constant and the first term is the result of Fermi liquid. Notice that this result is in disagreement with a result based on a renormalization group approach obtained in Ref. [24], even near $\eta = 1$. For HTSC $\eta = 2$ and $\frac{1}{T_1 T} \propto 1 - A\, T^{7/3}$. Note that the non-analytic correction is very small so that the Fermi liquid form is preserved.

2) $q \to 0$ limit of the transverse polarization function indicates that the transport scattering rate Γ_{tr} (which determines the DC conductivity) scales as $\Gamma_{\text{tr}} \propto \Omega^{\frac{4}{1+\eta}}$ at low frequencies (see Eq. (2.38) for more details). This result can be also obtained from the coefficient of the term which is proportional to q^2 in $\text{Im } \Pi_{00}(\mathbf{q},\Omega)$, and the

relation Im $\Pi_{11}(q \to 0, \Omega) = \frac{\Omega^2}{v_F^2 q^2}$Im $\Pi_{00}(q \to 0, \Omega)$. This result exactly agrees with that obtained by a different approach [11]. Note that $\Gamma_{\rm tr} < \Omega$ for $1 < \eta \le 2$.

3) From Eq. (2.4), one can see that the gauge field corrections are smaller than the result of free fermions along the curve $\Omega \propto q^{1+\eta}$ which is the dispersion relation of the gauge field. Therefore, the gauge field propagator is not renormalized. As mentioned above, non-renormalization of the gauge field propagator was first discussed in Ref. [20] within a self-consistent argument.

4) For $\eta \le 2$, the gauge field corrections to the polarization functions are less singular than the result of the free fermions for $\Omega < v_F q$. In particular, the edge of the particle-hole continuum in Im Π_{11} and Im Π_{00} still occurs at $\Omega \approx \tilde{v}_F q$, where $\tilde{v}_F$ is finite and shifted from the bare fermi velocity as in the usual Fermi liquid theory. We conclude that the two-particle Green's functions are consistent with those of a Fermi-liquid with a finite effective mass. However, a combination of a divergent mass and divergent Fermi-liquid parameters cannot be ruled out.

The remainder of this chapter is organized as the following. In section 2.2, the transverse polarization function for $q \to 0$ case is calculated. The cancellation of anomalous terms (coming from the self energy and the vertex correction) up to $(1/N)^0$th order is explicitly shown (where N is the number of species of fermions). We also discuss the optical conductivity using the information of the calculated transverse polarization function. In section 2.3, we calculate the transverse polarization function for finite $q \ll k_F$ case. It is also argued that the gauge field propagator is not renormalized up to two-loop order. In section 2.4, the density-density correlation function is calculated up to two-loop order for finite $q \ll k_F$. In section 2.5, the results are compared to the conventional Fermi-liquid theory and their implication is discussed. We conclude this chapter in section 2.6.

2.2 The Transverse Polarization Function for $q \to 0$ and Optical Conductivity

Let us consider a large N generalized model, where N is the number of species of fermions. In this model, each fermion bubble carries a factor of N and each gauge field line gives a factor of $1/N$. Thus, for example, Π^0_{00} and Π^0_{11} obtained in the previous section should be multiplied by N.

In this section, we consider only the $q \to 0$ case of the transverse polarization function: $\Pi_{11}(\mathbf{q} \to 0, i\nu)$. However, the relevant diagrams are the same even for $q \neq 0$ case. The leading order contribution is Π^0_{11} which is proportional to N. The relevant diagrams in the next order (*i.e.* $(1/N)^0$th order) are given by Figure 2-1 (a)-(g). For convenience let us define the following quantities: $\Pi^{(1)}_{11}$ = (a) + (b) and $\Pi^{(2)}_{11}$ = (c) + (d). The formal expressions of these quantities for $q \to 0$ case are given by

$$\Pi^{(1)}_{11} = -\int \frac{d^2k}{(2\pi)^2} \frac{d\omega}{2\pi} \left[\frac{k^2 - (\mathbf{k}\cdot\hat{\mathbf{q}})^2}{m^2}\right] \Sigma(\mathbf{k}, i\omega)\ [G_0(\mathbf{k}, i\omega)]^2\ G_0(\mathbf{k}, i\omega + i\nu)\ , \quad (2.9)$$

and

$$\Pi_{11}^{(2)} = -\int \frac{d^2k}{(2\pi)^2} \frac{d\omega}{2\pi} \left[\frac{k^2 - (\mathbf{k}\cdot\hat{\mathbf{q}})^2}{m^2}\right] \Sigma(\mathbf{k}, i\omega + i\nu) \left[G_0(\mathbf{k}, i\omega + i\nu)\right]^2 G_0(\mathbf{k}, i\omega) . \tag{2.10}$$

These two equations can be rewritten as

$$\begin{aligned}\Pi_{11}^{(1)} &= -\int \frac{d^2k}{(2\pi)^2} \frac{d\omega}{2\pi} \left[\frac{k^2 - (\mathbf{k}\cdot\hat{\mathbf{q}})^2}{m^2}\right] \frac{\Sigma(\mathbf{k}, i\omega)}{i\nu} \\ &\quad \times \Big([G_0(\mathbf{k}, i\omega)]^2 - G_0(\mathbf{k}, i\omega)\, G_0(\mathbf{k}, i\omega + i\nu) \Big) ,\end{aligned} \tag{2.11}$$

$$\begin{aligned}\Pi_{11}^{(2)} &= \int \frac{d^2k}{(2\pi)^2} \frac{d\omega}{2\pi} \left[\frac{k^2 - (\mathbf{k}\cdot\hat{\mathbf{q}})^2}{m^2}\right] \frac{\Sigma(\mathbf{k}, i\omega + i\nu)}{i\nu} \\ &\quad \times \Big([G_0(\mathbf{k}, i\omega + i\nu)]^2 - G_0(\mathbf{k}, i\omega + i\nu)\, G_0(\mathbf{k}, i\omega) \Big) .\end{aligned} \tag{2.12}$$

If we add Eq. (2.11) and Eq. (2.12), the first terms in each polarization bubble are cancelled by each other and the remaning parts give us

$$\begin{aligned}\Pi_{11}^{(1)} + \Pi_{11}^{(2)} &= \int \frac{d^2k}{(2\pi)^2} \frac{d\omega}{2\pi} \left[\frac{k^2 - (\mathbf{k}\cdot\hat{\mathbf{q}})^2}{m^2}\right] \\ &\quad \times \frac{\Sigma(\mathbf{k}, i\omega) - \Sigma(\mathbf{k}, i\omega + i\nu)}{i\nu} G_0(\mathbf{k}, i\omega)\, G_0(\mathbf{k}, i\omega + i\nu) .\end{aligned} \tag{2.13}$$

From the above expression, it can be easily seen that the contributions from (b) and (d) are automatically cancelled because the self energy corrections in these diagrams are just the same constants.

Next we consider the diagram given in Figure 2-1 (e). Here we have to include the vertex correction for $q \to 0$ case (Figure 2-2):

$$\begin{aligned}&\Gamma_1(\mathbf{k}, \mathbf{q} \to 0; i\omega, i\nu) \\ &= \int \frac{d^2q'}{(2\pi)^2} \frac{d\nu'}{2\pi} \left(-\frac{k_1 + q_1'}{m}\right) \left[\frac{k^2 - (\mathbf{k}\cdot\hat{\mathbf{q}}')^2}{m^2}\right] \\ &\quad \times G_0(\mathbf{k} + \mathbf{q}', i\omega + i\nu')\, G_0(\mathbf{k} + \mathbf{q}', i\omega + i\nu' + i\nu)\, D_{11}(\mathbf{q}', i\nu') .\end{aligned} \tag{2.14}$$

Then $\Pi_{11}^{(3)}(\mathbf{q} \to 0, i\nu)$ can be written as

$$\begin{aligned}\Pi_{11}^{(3)} &= -\int \frac{d^2k}{(2\pi)^2} \frac{d\omega}{2\pi} \left[-\frac{k_1}{m}\right] \Gamma_1(\mathbf{k}, \mathbf{q} \to 0; i\omega, i\nu)\, G_0(\mathbf{k}, i\omega)\, G_0(\mathbf{k}, i\omega + i\nu) \\ &= \Pi_{11}^{(3,1)} + \Pi_{11}^{(3,2)} ,\end{aligned} \tag{2.15}$$

where

$$\Pi_{11}^{(3,1)} = -\int \frac{d^2k}{(2\pi)^2} \frac{d\omega}{2\pi} \left[\frac{k^2 - (\mathbf{k}\cdot\hat{\mathbf{q}})^2}{m^2}\right] G_0(\mathbf{k}, i\omega)\, G_0(\mathbf{k}, i\omega + i\nu)$$

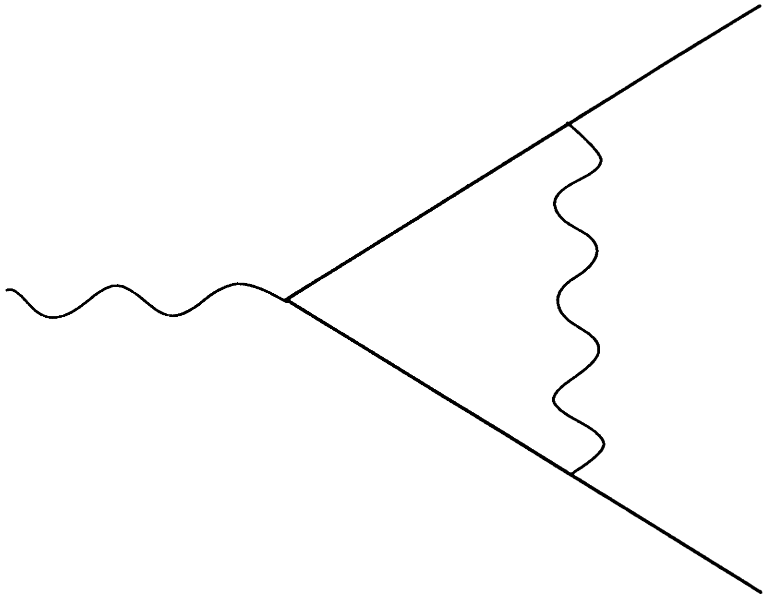


Figure 2-2: The diagram that corresponds to the lowest-order vertex correction $\Gamma_0(\mathbf{k},\mathbf{q},i\omega,i\nu)$ or $\Gamma_1(\mathbf{k},\mathbf{q},i\omega,i\nu)$.

$$
\begin{aligned}
& \times \int \frac{d^2q'}{(2\pi)^2} \frac{d\nu'}{2\pi} \left[\frac{k^2 - (\mathbf{k}\cdot\hat{\mathbf{q}}')^2}{m^2}\right] \\
& \times G_0(\mathbf{k}+\mathbf{q}', i\omega + i\nu')\, G_0(\mathbf{k}+\mathbf{q}', i\omega + i\nu' + i\nu)\, D_{11}(\mathbf{q}', i\nu')\ , \qquad (2.16)
\end{aligned}
$$

and

$$
\begin{aligned}
\Pi_{11}^{(3,2)} \; = \; & -\int \frac{d^2k}{(2\pi)^2} \frac{d\omega}{2\pi}\, G_0(\mathbf{k}, i\omega)\, G_0(\mathbf{k}, i\omega + i\nu) \\
& \times \int \frac{d^2q'}{(2\pi)^2} \frac{d\nu'}{2\pi} \left(\frac{q_1' k_1}{m^2}\right) \left[\frac{k^2 - (\mathbf{k}\cdot\hat{\mathbf{q}}')^2}{m^2}\right] \\
& \times G_0(\mathbf{k}+\mathbf{q}', i\omega + i\nu')\, G_0(\mathbf{k}+\mathbf{q}', i\omega + i\nu' + i\nu)\, D_{11}(\mathbf{q}', i\nu')\ . \qquad (2.17)
\end{aligned}
$$

Here we would like to point out that $\Pi_{11}^{(3,1)}$ is more singular than $\Pi_{11}^{(3,2)}$. This can be easily seen from the fact that $\Pi_{11}^{(3,2)}$ can be obtained by replacing $k_1^2/m^2 = \left[\frac{k^2-(\mathbf{k}\cdot\hat{\mathbf{q}})^2}{m^2}\right]$ in the integrand of Eq. (2.16) by $q_1' k_1/m^2$. Using $q_1' = q_{\parallel}'$ sin $\theta_{\mathbf{kq}} + q_{\perp}'$ cos $\theta_{\mathbf{kq}}$ and $\xi_{\mathbf{k}+\mathbf{q}} \approx \xi_{\mathbf{k}} + v_F q_{\parallel} + q_{\perp}^2/2m$, one can do the integrals over $q_{\parallel}'$ and $q_{\perp}'$ in Eq. (2.17). Since the contribution from $q_{\perp}'$ cos $\theta_{\mathbf{kq}}$ term becomes an odd function of $q_{\perp}'$, this term vanishes. By a formal manipulation, one can replace $q_{\parallel}'$ by $q_{\perp}'^2/k_F$ so that q_1' factor becomes effectively $(q_{\perp}'^2/k_F)$ sin $\theta_{\mathbf{kq}}$. Since the integrand is dominated by $|\nu| \sim (\chi/\gamma)\,|q_{\perp}|^{1+\eta}$ scaling given by the pole of the gauge field propagator, replacing k_1 by q_1' gives rise to an additional factor which is proportional to $|\nu|^{\frac{2}{1+\eta}}$. Therefore,

$\Pi_{11}^{(3,2)}$ should be less singular than $\Pi_{11}^{(3,1)}$ by the factor $|\nu|^{\frac{2}{1+\eta}}$ in the low frequency limit.

Note that $\Pi_{11}^{(3,1)}$ can be rewritten as

$$\Pi_{11}^{(3,1)} = -\int \frac{d^2k}{(2\pi)^2}\, \frac{d\omega}{2\pi} \left[\frac{k^2 - (\mathbf{k}\cdot\hat{\mathbf{q}})^2}{m^2}\right] \Gamma_0(\mathbf{k},\mathbf{q}\to 0; i\omega, i\nu) G_0(\mathbf{k}, i\omega)\ G_0(\mathbf{k}, i\omega + i\nu)\ , \tag{2.18}$$

where Γ_0 is the scalar vertex:

$$\begin{aligned} &\Gamma_0(\mathbf{k},\mathbf{q}; i\omega, i\nu) \\ =\ & \int \frac{d^2q'}{(2\pi)^2}\, \frac{d\nu'}{2\pi} \left[\frac{k^2 - (\mathbf{k}\cdot\hat{\mathbf{q}}')^2}{m^2}\right] \\ &\times G_0(\mathbf{k}+\mathbf{q}', i\omega + i\nu')\ G_0(\mathbf{k}+\mathbf{q}'+\mathbf{q}, i\omega + i\nu' + i\nu)\ D_{11}(\mathbf{q}', i\nu')\ . \end{aligned} \tag{2.19}$$

From the relation,

$$\begin{aligned} &\Sigma(\mathbf{k}, i\omega) - \Sigma(\mathbf{k}, i\omega + i\nu) \\ =\ & \int \frac{d^2q'}{(2\pi)^2}\, \frac{d\nu'}{2\pi} \left[\frac{k^2 - (\mathbf{k}\cdot\hat{\mathbf{q}}')^2}{m^2}\right] \\ &\times \left[G_0(\mathbf{k}+\mathbf{q}', i\omega + i\nu') - G_0(\mathbf{k}+\mathbf{q}', i\omega + i\nu' + i\nu)\right]\ D_{11}(\mathbf{q}', i\nu') \\ =\ & \int \frac{d^2q'}{(2\pi)^2}\, \frac{d\nu'}{2\pi} \left[\frac{k^2 - (\mathbf{k}\cdot\hat{\mathbf{q}}')^2}{m^2}\right] i\nu \\ &\times G_0(\mathbf{k}+\mathbf{q}', i\omega + i\nu')\ G_0(\mathbf{k}+\mathbf{q}', i\omega + i\nu' + i\nu)\ D_{11}(\mathbf{q}', i\nu')\ . \end{aligned} \tag{2.20}$$

we get the following identity:

$$\frac{\Sigma(\mathbf{k}, i\omega) - \Sigma(\mathbf{k}, i\omega + i\nu)}{i\nu} = \Gamma_0(\mathbf{k},\mathbf{q}\to 0; i\omega, i\nu)\ . \tag{2.21}$$

This is nothing but the Ward identity. From Eqs. (2.13), (2.18), and (2.21), we have

$$\Pi_{11}^{(1)} + \Pi_{11}^{(2)} + \Pi_{11}^{(3,1)} = 0\ . \tag{2.22}$$

Now the remaining piece is just $\Pi_{11}^{(3,2)}$. Following the procedures of integration mentioned above, in the low frequency limit, we get

$$\Pi_{11}^{(3,2)} \approx -\frac{1+\eta}{4\pi^2\ (5+\eta)\ \sin\left(\frac{3-\eta}{1+\eta}\pi\right)}\ \frac{v_F}{m}\ \frac{\gamma^{\frac{3-\eta}{1+\eta}}}{\chi^{\frac{4}{1+\eta}}}\ |\nu|^{\frac{3-\eta}{1+\eta}}\ . \tag{2.23}$$

Here it is worthwhile to compare this result with $\Pi_{11}^{(1)} + \Pi_{11}^{(2)}$ and $\Pi_{11}^{(3,1)}$, *i.e.*, the results before cancellation. By a straightforward calculation, one can get

$$\Pi_{11}^{(1)} + \Pi_{11}^{(2)} \approx -\frac{2\ (1+\eta)}{\pi\ (3+\eta)}\ m\ v_F^2\ \lambda\ |\nu|^{-\frac{\eta-1}{\eta+1}}\ . \tag{2.24}$$

In order to calculate $\Pi_{11}^{(3,1)}$, the vertex correction should be calculated. The vertex correction $\Gamma_0(\mathbf{k},\mathbf{q}\to 0; i\omega, i\nu)$ is found to be

$$\Gamma_0 \approx -\frac{v_F}{\gamma}\,\frac{1}{2\pi\,\sin\left(\frac{2\pi}{1+\eta}\right)}\,\frac{1}{\nu}\left[\left(\frac{|\omega|\gamma}{\chi}\right)^{\frac{2}{1+\eta}}\,\mathrm{sgn}(\omega) - \left(\frac{|\omega+\nu|\gamma}{\chi}\right)^{\frac{2}{1+\eta}}\,\mathrm{sgn}(\omega+\nu)\right]. \tag{2.25}$$

Using Eqs. (2.18) and (2.25), $\Pi_{11}^{(3,1)}$ can be calculated as

$$\Pi_{11}^{(3,1)} \approx \frac{m\,v_F^3}{2\pi^2\,\sin\left(\frac{2\pi}{1+\eta}\right)}\left(\frac{1+\eta}{3+\eta}\right)\frac{1}{\gamma^{\frac{\eta-1}{\eta+1}}\,\chi^{\frac{2}{1+\eta}}}\,|\nu|^{-\frac{\eta-1}{\eta+1}}\,. \tag{2.26}$$

Note that, as mentioned above, $\Pi_{11}^{(1)}+\Pi_{11}^{(2)}$ and $\Pi_{11}^{(3,1)}$ are more singular than $\Pi_{11}^{(3,2)}$ by $|\nu|^{-\frac{2}{1+\eta}}$ in the low frequency limit. The important point is that these singular terms are cancelled by each other due to the Ward identity.

Now let us look at the diagrams of (f) and (g). Let $\Pi_{11}^{(4)}$ = (f) and $\Pi_{11}^{(5)}$ = (g). The formal expressions of these diagrams for $q\to 0$ case are given by

$$\begin{aligned}\Pi_{11}^{(4)} &= \int \frac{d^2q'}{(2\pi)^2}\,\frac{d\nu'}{2\pi}\,\frac{d^2k'}{(2\pi)^2}\,\frac{d\omega'}{2\pi}\,\frac{d^2k''}{(2\pi)^2}\,\frac{d\omega''}{2\pi}\\ &\quad\times\left[\frac{\mathbf{k}'\cdot\mathbf{k}''-(\mathbf{k}'\cdot\hat{\mathbf{q}}')\,(\mathbf{k}''\cdot\hat{\mathbf{q}}')}{m^2}\right]^2\,D_{11}(\mathbf{q}',i\nu')\,D_{11}(\mathbf{q}',i\nu'+i\nu)\\ &\quad\times G_0(\mathbf{k}',i\omega')\,G_0(\mathbf{k}',i\omega'+i\nu)\,G_0(\mathbf{k}'-\mathbf{q}',i\omega'-i\nu')\\ &\quad\times G_0(\mathbf{k}'',i\omega'')\,G_0(\mathbf{k}'',i\omega''+i\nu)\,G_0(\mathbf{k}''-\mathbf{q}',i\omega''-i\nu')\,,\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}\Pi_{11}^{(5)} &= \int \frac{d^2q'}{(2\pi)^2}\,\frac{d\nu'}{2\pi}\,\frac{d^2k'}{(2\pi)^2}\,\frac{d\omega'}{2\pi}\,\frac{d^2k''}{(2\pi)^2}\,\frac{d\omega''}{2\pi}\\ &\quad\times\left[\frac{\mathbf{k}'\cdot\mathbf{k}''-(\mathbf{k}'\cdot\hat{\mathbf{q}}')\,(\mathbf{k}''\cdot\hat{\mathbf{q}}')}{m^2}\right]^2\,D_{11}(\mathbf{q}',i\nu')\,D_{11}(\mathbf{q}',i\nu'+i\nu)\\ &\quad\times G_0(\mathbf{k}',i\omega')\,G_0(\mathbf{k}',i\omega'+i\nu)\,G_0(\mathbf{k}'-\mathbf{q}',i\omega'-i\nu')\\ &\quad\times G_0(\mathbf{k}'',i\omega'')\,G_0(\mathbf{k}'',i\omega''+i\nu)\,G_0(\mathbf{k}''+\mathbf{q}',i\omega''+i\nu'+i\nu)\,.\end{aligned} \tag{2.28}$$

By changing variables as $\mathbf{q}'\to-\mathbf{q}'$, $\nu'\to-\nu'-\nu$ and using $D_{11}(-\mathbf{q}',-i\nu') = D_{11}(\mathbf{q}',i\nu')$, we get

$$\begin{aligned}\Pi_{11}^{(4)}+\Pi_{11}^{(5)} &= \frac{1}{2}\int\frac{d^2q'}{(2\pi)^2}\,\frac{d\nu'}{2\pi}\,D_{11}(\mathbf{q}',i\nu')\,D_{11}(\mathbf{q}',i\nu'+i\nu)\\ &\quad\times\left[\frac{d^2k}{(2\pi)^2}\,\frac{d\omega}{2\pi}\,\frac{k_1}{m}\left(\frac{k\,\sin\theta_{\mathbf{k}\mathbf{q}'}}{m}\right)^2 G_0(\mathbf{k},i\omega)\,G_0(\mathbf{k},i\omega+i\nu)\right.\\ &\quad\left.\times\left(G_0(\mathbf{k}+\mathbf{q}',i\omega+i\nu'+i\nu)+G_0(\mathbf{k}-\mathbf{q}',i\omega-i\nu')\right)\right]^2,\end{aligned} \tag{2.29}$$

where $\theta_{\mathbf{kq}'}$ is the angle between $\mathbf{k}$ and $\mathbf{q}'$. In the low frequency limit, we get

$$\Pi_{11}^{(4)} + \Pi_{11}^{(5)} \approx -c_1 \frac{v_F}{m} \frac{\gamma^{\frac{3-\eta}{1+\eta}}}{\chi^{\frac{4}{1+\eta}}} |\nu|^{\frac{3-\eta}{1+\eta}} , \tag{2.30}$$

where c_1 is a constant. One can also show that

$$\begin{aligned} \Pi_{11}^{(4)} &\approx -c_0 \frac{m\, v_F^3}{\gamma^{\frac{\eta-1}{\eta+1}} \chi^{\frac{2}{1+\eta}}} |\nu|^{-\frac{\eta-1}{\eta+1}} , \\ \Pi_{11}^{(5)} &\approx c_0 \frac{m\, v_F^3}{\gamma^{\frac{\eta-1}{\eta+1}} \chi^{\frac{2}{1+\eta}}} |\nu|^{-\frac{\eta-1}{\eta+1}} - c_1 \frac{v_F}{m} \frac{\gamma^{\frac{3-\eta}{1+\eta}}}{\chi^{\frac{4}{1+\eta}}} |\nu|^{\frac{3-\eta}{1+\eta}} , \end{aligned} \tag{2.31}$$

where c_0 is a constant. That is, there is also a cancellation between the singular parts of $\Pi_{11}^{(4)}$ and $\Pi_{11}^{(5)}$.

Gathering all the previous informations and using $\Pi_{11}^{0}(\mathbf{q} \to 0, i\nu) = \frac{Nn}{m}$, we can conclude that

$$\Pi_{11} \approx \frac{Nn}{m} - c_2 \frac{k_F}{m^2} \frac{\gamma^{\frac{3-\eta}{1+\eta}}}{\chi^{\frac{4}{1+\eta}}} |\nu|^{\frac{3-\eta}{1+\eta}} \tag{2.32}$$

up to $(1/N)^0$th order, where c_2 is a constant.

In order to calculate the optical conductivity, we have to consider the bubble diagrams with two external lines that represent the coupling to the external vector potential A_μ while the internal gauge field lines are due to a_μ. There are additional diagrams generated by $\psi^\dagger a_\mu A^\mu \psi$ vertex. All the additional diagrams except one (shown in Figure 2-3 (a)) vanish due to the symmetry of the integrand. A typical diagram which vanishes is shown in Figure 2-3 (b).

It turns out that the diagram represented by Figure 2-3 (a) gives an imaginary part which is higher order in frequency compared to $|\nu|^{\frac{3-\eta}{1+\eta}}$ so that it is irrelevant in the low freqency limit. Now we can use the imaginary part of the transverse polarization function in the Minkowski space $\Pi_{11}(\mathbf{q} \to 0, \Omega) = \Pi_{11}(\mathbf{q} \to 0, i\nu \to \Omega + i\delta)$ to calculate the real part of the optical conductivity:

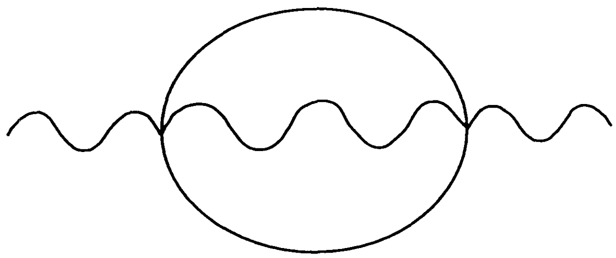
$$\text{Re } \sigma(\Omega) = -e^2 \frac{\text{Im } \Pi_{11}(\Omega)}{\Omega} . \tag{2.33}$$

From Eq. (2.32), Re $\sigma(\Omega)$ is given by

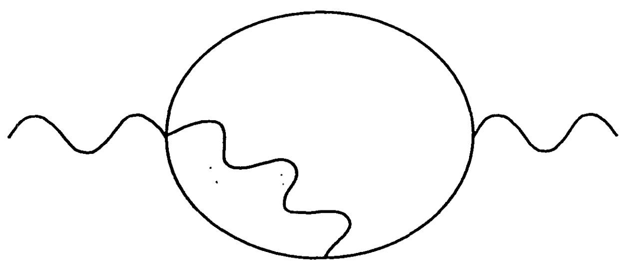
$$\text{Re } \sigma(\Omega) \propto \frac{e^2 k_F}{m^2} \frac{\gamma^{\frac{3-\eta}{1+\eta}}}{\chi^{\frac{4}{1+\eta}}} \Omega^{-2\left(\frac{\eta-1}{\eta+1}\right)} . \tag{2.34}$$

If there were no cancellation, the result would look quite different. For example, if we did not consider the vertex correction, the result from $\Pi_{11}^{(1)} + \Pi_{11}^{(2)}$ would be

$$\text{Re } \sigma_{nv}(\Omega) \propto \frac{e^2 m v_F^3}{\gamma^{\frac{\eta-1}{\eta+1}} \chi^{\frac{2}{1+\eta}}} \Omega^{-\frac{2\eta}{1+\eta}} , \tag{2.35}$$



(a)



(b)

Figure 2-3: (a) The non-vanishing diagram generated by $\psi^\dagger a_\mu A^\mu \psi$ vertex. (b) A typical vanishing diagram generated by $\psi^\dagger a_\mu A^\mu \psi$ vertex.

where σ_{nv} represents the conductivity without vertex correction.

Now we are going to show that the right answer given by Eq. (2.34) is consistent with a modified Drude formula if we assume that the transport scattering rate (which is the inverse of the transport time $\tau_{\rm tr}$) of the fermion is given by $\Gamma_{\rm tr}(\Omega) \propto \frac{1}{N} \frac{1}{mk_F} (\gamma^{\frac{3-\eta}{1+\eta}}/\chi^{\frac{4}{1+\eta}})\ \Omega^{\frac{4}{1+\eta}}$.

First of all, for later convenience, let us calculate the inverse of the transport time $\tau^0_{\rm tr}$ of the fermion [11] using the imaginary part of the self energy $\Sigma(\mathbf{k},\Omega)$. For this purpose, we can just include the factor $1 - \cos\Theta = 2\sin^2(\Theta/2)$ in the integrand of the expression for Im $\Sigma(\mathbf{k},\Omega)$, where Θ is the angle between the wave vector of the fermion and that of the gauge field [11]. Using the fact that $\sin(\Theta/2) \approx q/2k_F$ and $q \sim \left(\frac{\gamma\Omega}{\chi}\right)^{\frac{1}{1+\eta}}$ inside the integral [11], we get

$$\frac{1}{\tau^0_{\rm tr}} \propto \frac{1}{N}\frac{1}{mk_F}\frac{\gamma^{\frac{3-\eta}{1+\eta}}}{\chi^{\frac{4}{1+\eta}}}\ \Omega^{\frac{4}{1+\eta}} \tag{2.36}$$

Therefore, we will essentially show that our result of the optical conductivity is consistent with the identification of $\Gamma_{\rm tr} = 1/\tau^0_{\rm tr}$ or $\tau_{\rm tr} = \tau^0_{\rm tr}$ in a modified Drude formula.

The Drude formula that is appropriate to the large N generalized model is given by

$$\text{Re}\ \sigma(\Omega) = \frac{Nne^2}{m}\ \frac{\Gamma_{\rm tr}}{\Omega^2 + \Gamma^2_{\rm tr}}\ . \tag{2.37}$$

In the large N limit, if we assume $\Gamma_{\rm tr} = 1/\tau^0_{\rm tr} \propto 1/N$,

$$\text{Re}\ \sigma(\Omega) \approx \frac{Nne^2}{m}\ \frac{\Gamma_{\rm tr}}{\Omega^2} \propto \frac{e^2 v_F}{m}\ \frac{\gamma^{\frac{3-\eta}{1+\eta}}}{\chi^{\frac{4}{1+\eta}}}\ \Omega^{-2(\frac{\eta-1}{\eta+1})}\ . \tag{2.38}$$

This is the same result as that of Eq. (2.34). The result of Eq. (2.35) can be reproduced in the same way if we assume that $\Gamma_{\rm tr}(\Omega) \propto \frac{1}{N}\ (mv_F^3)(\gamma^{-\frac{\eta-1}{\eta+1}}\ \chi^{-\frac{2}{1+\eta}})\ \Omega^{\frac{2}{1+\eta}}$ which is essentially the imaginary part of the self energy $\Sigma(\mathbf{k},\Omega)$. Therefore, the optical conductivity is consistent with the choice of $1/\tau^0_{\rm tr}$ rather than just the naive scattering rate (given by the self energy) as the transport scattering rate. Since the singular contribution, which gives Eq. (2.35), is cancelled by the vertex correction, we can again say that the leading singular behaviors of one-particle properties do not show up in the optical conductivity.

For finite temperature, one can replace Ω by T in $\Gamma_{\rm tr}$. Note that the DC-limit of the optical conductivity Re $\sigma(\Omega \to 0) = \frac{Nne^2}{m}\frac{1}{\Gamma_{\rm tr}}$ cannot be obtained by the $1/N$ expansion. However, one can infer the DC-limit by assuming that the full Re $\sigma(\Omega)$ is given by Eq. (2.37) (with $\Gamma_{\rm tr} = \Gamma_{\rm tr}(T)$) which is consistent with the result of the large-N limit of the optical conductivity. If $\Gamma_{\rm tr} \propto T^{\frac{4}{1+\eta}}$ was used, one would get Re $\sigma(T) \propto T^{-\frac{4}{1+\eta}}$ [11]. One the other hand, one would get Re $\sigma_{nv}(T) \propto T^{-\frac{2}{1+\eta}}$ if $\Gamma_{\rm tr} \propto T^{\frac{2}{1+\eta}}$ was used. In Ref. [21], the authors concluded that the resistivity of the system is proportional to $T^{2/3}$ for the short-range interaction ($\eta = 2$) and this is consistent with the latter case. Therefore, our result is in disagreement with their conclusion about the resistivity.

2.3 The Transverse Polarization Function for Finite $q \ll k_F$ and Non-Renormalization of the Gauge Field Propagator

It is not easy to find the polarization function for arbitrary $\mathbf{q}$ and ν. However, some simplifications can be made for $q \ll k_F$ case. In this section, we calculate $\Pi_{11}(\mathbf{q}, i\nu)$ for finite $q \ll k_F$ up to two-loop order. We set $N = 1$ first, and discuss the extension to the large-N case later.

First of all, $\Pi_{11}^{(1)}$ and $\Pi_{11}^{(2)}$ for finite $\mathbf{q}$ have the following form:

$$
\begin{aligned}
\Pi_{11}^{(1)} &= -\int \frac{d^2k}{(2\pi)^2} \frac{d\omega}{2\pi} \left[\frac{k^2 - (\mathbf{k}\cdot\hat{\mathbf{q}})^2}{m^2}\right] \Sigma(\mathbf{k}, i\omega)\, [G_0(\mathbf{k}, i\omega)]^2\, G_0(\mathbf{k}+\mathbf{q}, i\omega + i\nu) \ , \\
\Pi_{11}^{(2)} &= -\int \frac{d^2k}{(2\pi)^2} \frac{d\omega}{2\pi} \left[\frac{k^2 - (\mathbf{k}\cdot\hat{\mathbf{q}})^2}{m^2}\right] \Sigma(\mathbf{k}+\mathbf{q}, i\omega + i\nu) \\
&\quad \times [G_0(\mathbf{k}+\mathbf{q}, i\omega + i\nu)]^2\, G_0(\mathbf{k}, i\omega) \ . \qquad (2.39)
\end{aligned}
$$

Using the similar method as that used in section 2.2, one can obtain

$$
\begin{aligned}
\Pi_{11}^{(1)} + \Pi_{11}^{(2)} &\approx \int \frac{d^2k}{(2\pi)^2} \frac{d\omega}{2\pi} \left[\frac{k^2 - (\mathbf{k}\cdot\hat{\mathbf{q}})^2}{m^2}\right] G_0(\mathbf{k}, i\omega)\, G_0(\mathbf{k}+\mathbf{q}, i\omega + i\nu) \\
&\quad \times \frac{\Sigma(\mathbf{k}, i\omega) - \Sigma(\mathbf{k}+\mathbf{q}, i\omega + i\nu)}{i\nu - v_F q \cos\theta_{\mathbf{kq}}} \ . \qquad (2.40)
\end{aligned}
$$

Next we should consider the vertex correction (Figure 2-2) for finite $\mathbf{q}$:

$$
\begin{aligned}
& \Gamma_1(\mathbf{k}, \mathbf{q}; i\omega, i\nu) \\
&= \int \frac{d^2q'}{(2\pi)^2} \frac{d\nu'}{2\pi} A(\mathbf{k}, \mathbf{q}, \mathbf{q}')\, B(\mathbf{k}, \mathbf{q}, \mathbf{q}') \\
&\quad \times G_0(\mathbf{k}+\mathbf{q}', i\omega + i\nu')\, G_0(\mathbf{k}+\mathbf{q}'+\mathbf{q}, i\omega + i\nu' + i\nu)\, D_{11}(\mathbf{q}', i\nu') \ , \qquad (2.41)
\end{aligned}
$$

where

$$
\begin{aligned}
A &= -\frac{k_1 + q_1' + q_1/2}{m} = -\frac{k_1 + q_1'}{m} \\
B &= \frac{1}{m}\left[(\mathbf{k}+\mathbf{q}'/2)\cdot(\mathbf{k}+\mathbf{q}+\mathbf{q}'/2) - (\mathbf{k}+\mathbf{q}'/2)\cdot\hat{\mathbf{q}}'\ (\mathbf{k}+\mathbf{q}+\mathbf{q}'/2)\cdot\hat{\mathbf{q}}'\right] \qquad (2.42)
\end{aligned}
$$

For $q \ll k_F$ and $|\mathbf{k}| \approx k_F$, the following approximation can be made

$$
B \approx \frac{k^2 - (\mathbf{k}\cdot\hat{\mathbf{q}}')^2}{m} \ . \qquad (2.43)
$$

Using this approximation, one can show that

$$\begin{aligned}
\Pi_{11}^{(3)} &= -\int \frac{d^2k}{(2\pi)^2}\,\frac{d\omega}{2\pi}\,\left[-\frac{k_1}{m}\right]\,\Gamma_1(\mathbf{k},\mathbf{q};i\omega,i\nu)\;G_0(\mathbf{k},i\omega)\;G_0(\mathbf{k}+\mathbf{q},i\omega+i\nu)\\
&\approx \Pi_{11}^{(3,3)}+\Pi_{11}^{(3,4)}\;, \qquad (2.44)
\end{aligned}$$

where

$$\begin{aligned}
\Pi_{11}^{(3,3)} &= -\int \frac{d^2k}{(2\pi)^2}\,\frac{d\omega}{2\pi}\,\left[\frac{k^2-(\mathbf{k}\cdot\hat{\mathbf{q}})^2}{m^2}\right]\,\Gamma_0(\mathbf{k},\mathbf{q};i\omega,i\nu)\;G_0(\mathbf{k},i\omega)\;G_0(\mathbf{k}+\mathbf{q},i\omega+i\nu)\;,\\
\Pi_{11}^{(3,4)} &= -\int \frac{d^2k}{(2\pi)^2}\,\frac{d\omega}{2\pi}\,G_0(\mathbf{k},i\omega)\;G_0(\mathbf{k}+\mathbf{q},i\omega+i\nu)\\
&\quad\times\int \frac{d^2q'}{(2\pi)^2}\,\frac{d\nu'}{2\pi}\,\left(\frac{q_1'k_1}{m^2}\right)\,\left[\frac{k^2-(\mathbf{k}\cdot\hat{\mathbf{q}}')^2}{m^2}\right]\\
&\quad\times G_0(\mathbf{k}+\mathbf{q}',i\omega+i\nu')\;G_0(\mathbf{k}+\mathbf{q}'+\mathbf{q},i\omega+i\nu'+i\nu)\;D_{11}(\mathbf{q}',i\nu')\;. \qquad (2.45)
\end{aligned}$$

First, let us calculate the scalar vertex part $\Gamma_0(\mathbf{k},\mathbf{q};i\omega,i\nu)$. We use $\xi_{\mathbf{k}+\mathbf{q}'}\approx\xi_{\mathbf{k}}+v_Fq_{\|}'+q_{\perp}'^2/2m$ and $\xi_{\mathbf{k}+\mathbf{q}'+\mathbf{q}}\approx\xi_{\mathbf{k}}+v_Fq_{\|}'+v_Fq\cos\theta_{\mathbf{kq}}+\frac{qq_{\perp}'}{m}\sin\theta_{\mathbf{kq}}+q_{\perp}'^2/2m$ (where $q_{\|}'=q'\cos\theta_{\mathbf{kq}'}$ and $q_{\perp}'=q'\sin\theta_{\mathbf{kq}'}$) to perform the integral in Eq. (2.19). Using the fact that the important region of q' is the order of $\nu^{\frac{1}{1+\eta}}\ll 1$ so that $q'/k\approx q'/k_F\ll 1$, we conclude [19, 22, 20] that $q_{\|}'/k_F\approx(q_{\perp}'/k_F)^2$ and we can approximate the gauge field propagator as $D_{11}(\mathbf{q}',i\nu')\approx 1/(\gamma|\nu'|/|q_{\perp}'|+\chi|q_{\perp}'|^{\eta})$. After performing $q_{\|}'$ integral, we get

$$\begin{aligned}
\Gamma_0(\mathbf{k},\mathbf{q};i\omega,i\nu) &\approx -iv_F\int\frac{d\nu'}{2\pi}\int\frac{dq_{\perp}'}{2\pi}\,(\mathrm{sgn}(\omega+\nu')-\mathrm{sgn}(\omega+\nu+\nu'))\\
&\quad\times\frac{1}{i\nu-v_Fq\cos\theta_{\mathbf{kq}}-\frac{qq_{\perp}'}{m}\sin\theta_{\mathbf{kq}}}\,\frac{1}{\gamma\frac{|\nu'|}{|q_{\perp}'|}+\chi|q_{\perp}'|^{\eta}}\;. \qquad (2.46)
\end{aligned}$$

Now ν' integral gives

$$\begin{aligned}
&\Gamma_0(\mathbf{k},\mathbf{q};i\omega,i\nu)\\
\approx\;& -\frac{v_F}{\gamma}\,\frac{1}{\pi^2}\int_{-k_F}^{k_F}dq_{\perp}'\,\frac{|q_{\perp}'|}{\nu+iv_Fq\cos\theta_{\mathbf{kq}}+i\frac{qq_{\perp}'}{m}\sin\theta_{\mathbf{kq}}}\\
&\times\left[\ln\left(1+\frac{|\omega|\gamma}{|q_{\perp}'|^{1+\eta}\chi}\right)\mathrm{sgn}(\omega)-\ln\left(1+\frac{|\omega+\nu|\gamma}{|q_{\perp}'|^{1+\eta}\chi}\right)\mathrm{sgn}(\omega+\nu)\right]\;. \qquad (2.47)
\end{aligned}$$

By changing variables, one can get the following formula.

$$\begin{aligned}
&\Gamma_0(\mathbf{k},\mathbf{q};i\omega,i\nu)\\
\approx\;& -\frac{v_F}{\gamma}\,\frac{1}{\pi^2}\,\frac{1}{\nu+iv_Fq\cos\theta_{\mathbf{kq}}}\\
&\times\left[\left(\frac{|\omega|\gamma}{\chi}\right)^{\frac{2}{1+\eta}}F\left(\omega,\,\frac{(q/m)\sin\theta_{\mathbf{kq}}}{v_Fq\cos\theta_{\mathbf{kq}}-i\nu}\,\left[\frac{|\omega|\gamma}{\chi}\right]^{\frac{1}{1+\eta}}\right)\,\mathrm{sgn}(\omega)\right.
\end{aligned}$$

$$
- \left(\frac{|\omega+\nu|\gamma}{\chi}\right)^{\frac{2}{1+\eta}} F\left(\omega+\nu,\ \frac{(q/m)\,\sin\,\theta_{\mathbf{kq}}}{v_F q\,\cos\,\theta_{\mathbf{kq}} - i\nu}\left[\frac{|\omega+\nu|\gamma}{\chi}\right]^{\frac{1}{1+\eta}}\right)
$$
$$
\times\ \mathrm{sgn}(\omega+\nu)\Bigg] . \tag{2.48}
$$

Here $F(\omega, x)$ is defined as

$$
F(\omega, x) = \int_{-y_c}^{y_c} dy\ |y|\ \frac{\ln\,(1+|y|^{-1-\eta})}{1+xy}\ , \tag{2.49}
$$

where $y_c = k_F \left(\frac{\chi}{|\omega|\gamma}\right)^{\frac{1}{1+\eta}}$. It can be easily shown that $q \to 0$ limit of Eq. (2.48) is given by Eq. (2.25). On the other hand, the self energy can be rewritten as

$$
\Sigma(\mathbf{k}, \omega) \approx -i\frac{v_F}{\pi^2\gamma}\left(\frac{|\omega|\gamma}{\chi}\right)^{\frac{2}{1+\eta}} \mathrm{sgn}(\omega)\ F(\omega, 0) \tag{2.50}
$$

Collecting these results, it can be shown that

$$
\begin{aligned}
\Pi_{11}^{(1)} + \Pi_{11}^{(2)} + \Pi_{11}^{(3,3)} \approx & -\int \frac{d^2k}{(2\pi)^2}\frac{d\omega}{2\pi}\left[\frac{k^2-(\mathbf{k}\cdot\hat{\mathbf{q}})^2}{m^2}\right] G_0(\mathbf{k}, i\omega)\ G_0(\mathbf{k}+\mathbf{q}, i\omega+i\nu) \\
& \times\frac{iv_F}{\pi^2\gamma}\ \frac{1}{v_F q\,\cos\,\theta_{\mathbf{kq}} - i\nu}\left[I(\omega) - I(\omega+\nu)\right] ,
\end{aligned} \tag{2.51}
$$

where

$$
\begin{aligned}
I(\omega) = & \left(\frac{|\omega|\gamma}{\chi}\right)^{\frac{2}{1+\eta}} \mathrm{sgn}(\omega) \\
& \times\left[F\left(\omega,\ \frac{(q/m)\,\sin\,\theta_{\mathbf{kq}}}{v_F q\,\cos\,\theta_{\mathbf{kq}} - i\nu}\left[\frac{|\omega|\gamma}{\chi}\right]^{\frac{1}{1+\eta}}\right) - F(\omega, 0)\right] .
\end{aligned} \tag{2.52}
$$

The integrals in Eq. (2.51) can be evaluated as the following. Using $\int d^2k/(2\pi)^2 = (m/2\pi)\int d\xi_{\mathbf{k}}\ \int d\theta_{\mathbf{kq}}/2\pi$, one can perform $\xi_{\mathbf{k}}$ integral easily. The angular integral over $\theta_{\mathbf{kq}}$ can be done by contour integration, which requires long algebraic manipulations. The remaining ω integral and the y integral in $I(\omega)$ of Eq. (2.52) can be evaluated by scaling the integration variables and expanding the integrand in some limits. More details of the calculation will be demonstrated in the later evalution of the density-density correlation function (see the discussions about Eqs. (2.61)- (2.63) in section 2.4) which can be more easily calculated. First, for $|\nu| \ll v_F q$,

$$
\Pi_{11}^{(1)} + \Pi_{11}^{(2)} + \Pi_{11}^{(3,3)} \approx c_3\ \frac{m^2 v_F^3}{\gamma}\ \frac{|\nu|}{v_F q}\ \frac{(\gamma|\nu|/\chi)^{\frac{4}{1+\eta}}}{k_F^3 q}\ , \tag{2.53}
$$

while, in the other limit $|\nu| \gg v_F q$, we get

$$\Pi_{11}^{(1)} + \Pi_{11}^{(2)} + \Pi_{11}^{(3,3)} \approx c_4 \, \frac{m^2 v_F^3}{\gamma} \, \frac{q v_F}{|\nu|} \, \frac{q}{k_F} \left[\frac{(\gamma/\chi)^{\frac{2}{1+\eta}}}{m|\nu|^{\frac{\eta-1}{\eta+1}}} \right]^2 , \tag{2.54}$$

where c_3 and c_4 are dimensionless constants.

The calculation of $\Pi_{11}^{(3,4)}$ can be also done by the similar method used in the evaluation of $\Pi_{11}^{(3,3)}$. First, for $|\nu| \ll v_F q$, we get

$$\Pi_{11}^{(3,4)} \approx -\frac{m^2 v_F^3}{\gamma} \, \frac{|\nu|}{v_F q} \left[c_5 \, \frac{(\gamma|\nu|/\chi)^{\frac{2}{1+\eta}}}{k_F^2} + c_6 \, \frac{(\gamma|\nu|/\chi)^{\frac{3}{1+\eta}}}{k_F^2 q} \right] , \tag{2.55}$$

whereas, in the other limit $|\nu| \gg v_F q$,

$$\Pi_{11}^{(3,4)} \approx -\frac{1+\eta}{4\pi^2(5+\eta)} \, \frac{1}{\sin\left(\frac{4\pi}{1+\eta}\right)} \, \frac{v_F}{m} \, \frac{\gamma^{\frac{3-\eta}{1+\eta}}}{\chi^{\frac{4}{1+\eta}}} \, |\nu|^{\frac{3-\eta}{1+\eta}} - c_7 \, \frac{m^2 v_F^3}{\gamma} \, \frac{v_F q^2}{m^2 (\chi/\gamma)^{\frac{3}{1+\eta}} |\nu|^{\frac{3}{1+\eta}}} , \tag{2.56}$$

where c_5, c_6 and c_7 are dimensionless constants.

From the above results, it can be shown that $|\Pi_{11}^{(1)} + \Pi_{11}^{(2)} + \Pi_{11}^{(3,3)}| < |\Pi_{11}^{(3,4)}|$ for relevant limits. Therefore, the imaginary part of the transverse polarization function $\Pi_{11}(\mathbf{q}, \Omega)$ (in the Minkowski space) up to two-loop order is given by the following formulae. For $\Omega \ll v_F q$, we get

$$\mathrm{Im}\, \Pi_{11}(\mathbf{q}, \Omega) \approx -\frac{m v_F^2}{2\pi} \, \frac{\Omega}{v_F q} \left[1 - a \, \frac{m v_F}{\gamma} \, \frac{(\gamma\Omega/\chi)^{\frac{2}{1+\eta}}}{k_F^2} - b \, \frac{m v_F}{\gamma} \, \frac{(\gamma\Omega/\chi)^{\frac{3}{1+\eta}}}{k_F^2 q} \right] , \tag{2.57}$$

where a and b are dimensionless constants. Note that the correction is small as far as $1 < \eta \leq 2$ is concerned. On the other hand, for $\Omega \gg v_F q$, we have

$$\mathrm{Im}\, \Pi_{11}(\mathbf{q}, \Omega) \approx -\frac{1+\eta}{8\pi^2(5+\eta)} \, \frac{1}{\sin\left(\frac{2\pi}{1+\eta}\right)} \, \frac{v_F}{m} \, \frac{\gamma^{\frac{3-\eta}{1+\eta}}}{\chi^{\frac{4}{1+\eta}}} \, \Omega^{\frac{3-\eta}{1+\eta}} \left[1 + c \, m v_F^3 \left(\frac{\chi}{\gamma} \right)^{\frac{1}{1+\eta}} \frac{q^2}{\Omega^{\frac{2\eta+3}{\eta+1}}} \right] , \tag{2.58}$$

where c is a dimensionless constant.

For $\Omega > v_F q$, there is no contribution to Im Π_{11} from the free fermion bubble because the regime is outside the particle-hole continuum. Therefore, any non-zero contribution to Im Π_{11} for $\Omega \gg v_F q$ entirely comes from the gauge field correction. Note that the first term in Eq.(65) dominates for $\Omega > (m v_F^3)^{\frac{1+\eta}{2\eta+3}} (\chi/\gamma)^{\frac{1}{2\eta+3}} q^{\frac{2\eta+2}{2\eta+3}}$. On the other hand, the second term becomes more important for $v_F q \ll \Omega < (m v_F^3)^{\frac{1+\eta}{2\eta+3}} (\chi/\gamma)^{\frac{1}{2\eta+3}} q^{\frac{2\eta+2}{2\eta+3}}$ so that Im $\Pi_{11} \propto v_F^4 \frac{\gamma^{\frac{2-\eta}{1+\eta}}}{\chi^{\frac{3}{1+\eta}}} \frac{q^2}{\Omega^{\frac{3\eta}{1+\eta}}}$ in this regime. As we approach the line given by $\Omega = v_F q$, Im Π_{11} becomes $v_F^{\frac{4+\eta}{1+\eta}} \frac{\gamma^{\frac{2-\eta}{1+\eta}}}{\chi^{\frac{3}{1+\eta}}} q^{\frac{2-\eta}{1+\eta}}$ as a function of q.

In the case of $\Omega \ll v_F q$, the free fermion bubble gives Im $\Pi_{11}^0 = -\frac{m v_F^2}{2\pi} \frac{\Omega}{v_F q}$.

Note that $\text{Im}\,\Pi_{11}(\mathbf{q},\Omega) \approx -\frac{mv_F^2}{2\pi}\frac{\Omega}{v_F q}\left[1 - a\,\frac{mv_F}{\gamma}\,\frac{(\gamma\Omega/\chi)^{\frac{2}{1+\eta}}}{k_F^2}\right]$ for $\Omega < (\chi/\gamma)q^{1+\eta}$ and $\text{Im}\,\Pi_{11} \approx -\frac{mv_F^2}{2\pi}\frac{\Omega}{v_F q}\left[1 - b\,\frac{mv_F}{\gamma}\,\frac{(\gamma\Omega/\chi)^{\frac{3}{1+\eta}}}{k_F^2 q}\right]$ for $(\chi/\gamma)q^{1+\eta} < \Omega \ll v_F q$. It is gratifying to note that, along the line $\Omega = v_F q$, the correction to $\text{Im}\,\Pi_{11}$ given by the above expression agrees in its q dependence with that obtained by approaching from $\Omega \gg v_F q$ given in the last paragraph. In any case, the corrections are small compared to the free fermion result for $1 < \eta \leq 2$.

Using the result of Π_{11} for $|\nu| \ll v_F q$, we can discuss the issue of the renormalization of the gauge field propagator. Recall that the dispersion relation of the gauge field obtained from the one-loop correction is given by $|\nu| \sim (\chi/\gamma)q^{1+\eta}$ [6, 11, 12], which is below the line of $|\nu| = v_F q$ for sufficiently small q. Along the line of $|\nu| \sim (\chi/\gamma)q^{1+\eta}$, one can easily see that the correction to Π^0_{11} is smaller by $\frac{mv_F}{\gamma}\left(\frac{q}{k_F}\right)^2$. Therefore, the gauge field propagator is not renormalized up to two-loop order. As mentioned in the introduction, non-renormalization of the gauge field propagator was first discussed by Polchinski within a self-consistent argument and without vertex correction. In Ref. [21], the authors discussed the relevance of $\Gamma^{(3)}(a_\mu)$ and $\Gamma^{(4)}(a_\mu)$, which are coefficients of the a^3 and a^4 terms in the expansion of the effective action of the gauge field. They concluded that $\Gamma^{(3)}(a_\mu)$ and $\Gamma^{(4)}(a_\mu)$ are irrelevant so that the gauge field is not renormalized. Since the two-loop diagrams we considered are generated from $\Gamma^{(4)}(a_\mu)$, our calculation is consistent with their conclusion. By analogy, we expect that $\Pi^{(4)}_{11}$ and $\Pi^{(5)}_{11}$ are irrelevant for the renormalization of the gauge field because these are generated from $\Gamma^{(3)}(a_\mu)$. We also directly evaluated $\Gamma^{(3)}(a_\mu)$ and confirmed the argument of Ref. [21]. Therefore, one can expect that the gauge field is not renormalized up to $(1/N)^0$th order in the $1/N$ expansion. That is, the RPA calculation gives the leading contributions in the low energy limit.

2.4 The Density-Density Correlation Function for Finite $q \ll k_F$

The polarization function for the density channel $\Pi_{00}(\mathbf{q},\Omega)$ can be also calculated in a similar way as used in section 2.3. In this section, we consider the two-loop corrections given by Figure 2-1 (a)-(e) and finite $q \ll k_F$ case. The sum of the contributions from the self-energy corrections given by Figure 2-1 (a)-(d) can be written as

$$\Pi^{(1)}_{00} \approx \int \frac{d^2k}{(2\pi)^2}\frac{d\omega}{2\pi}\, G_0(\mathbf{k},i\omega)\, G_0(\mathbf{k}+\mathbf{q},i\omega+i\nu)\,\frac{\Sigma(\mathbf{k},i\omega)-\Sigma(\mathbf{k}+\mathbf{q},i\omega+i\nu)}{i\nu - v_F q\cos\theta_{\mathbf{kq}}}\,, \tag{2.59}$$

while the contribution given by Figure 2-1 (e), which comes from the vertex correction, can be also written as

$$\Pi^{(2)}_{00} = -\int \frac{d^2k}{(2\pi)^2}\frac{d\omega}{2\pi}\,\Gamma_0(\mathbf{k},\mathbf{q};i\omega,i\nu)\, G_0(\mathbf{k},i\omega)\, G_0(\mathbf{k}+\mathbf{q},i\omega+i\nu)\,. \tag{2.60}$$

Using Eqs. (2.48) and (4.25), it can be shown that

$$\begin{aligned}\Pi_{00}^{(1)}+\Pi_{00}^{(2)} &\approx -\int \frac{d^2k}{(2\pi)^2}\,\frac{d\omega}{2\pi}\,G_0(\mathbf{k},i\omega)\,G_0(\mathbf{k}+\mathbf{q},i\omega+i\nu)\\ &\quad\times\frac{iv_F}{\pi^2\gamma}\,\frac{1}{v_Fq\,\cos\theta_{\mathbf{kq}}-i\nu}\left[I(\omega)-I(\omega+\nu)\right],\end{aligned} \tag{2.61}$$

where $I(\omega)$ is given by Eq. (2.52). Using $\int d^2k/(2\pi)^2 = (m/2\pi)\int d\xi_{\mathbf{k}}\ \int d\theta_{\mathbf{kq}}/2\pi$, one can easily perform $\xi_{\mathbf{k}}$ integral, which generates the additional factor $v_Fq\cos\theta_{\mathbf{kq}}-i\nu$ in the denomenator of the integrand of Eq. (2.61). Recalling that $I(\omega)$ also has an angle dependence $\theta_{\mathbf{kq}}$, one can perform the angular integral over $\theta_{\mathbf{kq}}$ by contour integration, which requires long algebraic manipulations. After rescaling the ω integral by a new variable x and the y integral in $I(\omega)$ (see Eqs. (2.49) and (2.52)) by newly defined y, we get

$$\begin{aligned}\Pi_{00}^{(1)}+\Pi_{00}^{(2)} &\approx \frac{2k_F^3}{\pi^3\gamma}\frac{|\nu|}{v_F^2q^2}\int_0^1 dx\int_0^1 dy\ y\ \ln\left(1+\frac{x\beta^{1+\eta}}{y^{1+\eta}}\right)\\ &\quad\times\left[\frac{|\alpha|}{(1+\alpha^2)\sqrt{1+\alpha^2+y^2}}-\frac{|\alpha|}{(1+\alpha^2)^{3/2}}\right],\end{aligned} \tag{2.62}$$

where $\alpha=\frac{\nu}{v_Fq}$ and $\beta=\frac{1}{k_F}\left(\frac{|\nu|\gamma}{\chi}\right)^{\frac{1}{1+\eta}}$. In the small frequency ν limit, the parameter integrals can be done, yielding

$$\begin{aligned}\Pi_{00}^{(1)}+\Pi_{00}^{(2)} &\approx -\frac{a_1}{k_F^{\eta-2}\chi}\,\frac{|\alpha|^3}{(1+\alpha^2)^{3/2}}\\ &\quad-\frac{1+\eta}{4\pi^2(5+\eta)}\frac{1}{\sin\left(\frac{4\pi}{1+\eta}\right)}\frac{1}{k_F\gamma}\frac{1}{v_Fq}\left(\frac{\gamma|\nu|}{\chi}\right)^{\frac{4}{1+\eta}}\frac{\alpha^2}{(1+\alpha^2)^{5/2}},\end{aligned} \tag{2.63}$$

where a_1 is an undetermined constant. This formula is valid for all ratios of q and ν, as long as both are small. Note that the first term gives only an analytic contribution, which also arises in the usual Fermi liquid theory. Similar methods can be used to produce a somewhat more complicated formula valid for all α for the transverse polarization function Π_{11} (for example, Eqs. (2.45) and (2.51) can be evaluated by a similar method).

After dropping the analytic contribution, we combine the free fermion contribution and perform analytic continuation to get, for $\Omega \ll v_Fq$,

$$\mathrm{Im}\ \Pi_{00}(q,\Omega)\approx -\frac{m}{2\pi}\frac{\Omega}{v_Fq}\left[1-\frac{1+\eta}{4\pi(5+\eta)}\frac{1}{\cos\left(\frac{\eta-1}{\eta+1}\pi\right)}\frac{1}{k_Fm}\frac{\gamma^{\frac{3-\eta}{1+\eta}}}{\chi^{\frac{4}{1+\eta}}}\Omega^{\frac{3-\eta}{1+\eta}}\left(\frac{\Omega}{v_Fq}\right)^2\right], \tag{2.64}$$

and for $\Omega \gg v_Fq$,

$$\mathrm{Im}\ \Pi_{00}(q,\Omega)\approx -\frac{1+\eta}{8\pi^2(5+\eta)}\frac{1}{\sin\left(\frac{2\pi}{1+\eta}\right)}\frac{1}{k_F}\frac{\gamma^{\frac{3-\eta}{1+\eta}}}{\chi^{\frac{4}{1+\eta}}}\Omega^{\frac{3-\eta}{1+\eta}}\left(\frac{v_Fq}{\Omega}\right)^2. \tag{2.65}$$

Note that Im $\Pi_{11}(q \to 0, \Omega) = \frac{\Omega^2}{v_F^2 q^2}$Im $\Pi_{00}(q \to 0, \Omega)$ is satisfied. Therefore, both of Im $\Pi_{11}(q \to 0, \Omega)$ and Im $\Pi_{00}(q \to 0, \Omega)$ give the same answer for the optical conductivity given by Eq. (2.34).

2.5 Comparision to the Fermi Liquid Theory

In section 2.2, it was shown that the resulting conductivity is consistent with a modefied Drude formula. In this section, we try to fit this result to the Fermi liquid theory framework to extract informations about the Fermi liquid parameters and examine whether the gauge field induces some singular or divergent parameters. In the Fermi liquid theory, the conductivity for N species of fermions is given by [31]

$$\sigma(\Omega) = \frac{Ne^2}{m^*} \frac{\tau}{1 - i\Omega\tau(m/m^*)} \,, \tag{2.66}$$

or

$$\text{Re } \sigma(\Omega) = \frac{Ne^2}{m} \frac{\Gamma_{\text{tr}}}{\Omega^2 + \Gamma_{\text{tr}}^2} \,, \tag{2.67}$$

where $\Gamma_{\text{tr}} = \Gamma_{\text{sc}} \frac{m^*}{m}$, $\Gamma_{\text{sc}} = 1/\tau$ is the scattering rate and τ is the scattering time. Here m^* is the effective mass of the fermion. Using the fact $\Gamma_{\text{tr}} \propto 1/N$ in the large N limit, we get

$$\text{Re } \sigma(\Omega) \approx \frac{Ne^2}{m} \frac{\Gamma_{\text{tr}}}{\Omega^2} \,. \tag{2.68}$$

Comparing the above result with Eq. (2.34) which is a result of the $1/N$ expansion, we can again identify Γ_{tr} with $1/\tau_{\text{tr}}^0$ given in Eq. (2.36). Therefore, we can conclude that $\Gamma_{\text{tr}} = \Gamma_{\text{sc}} \frac{m^*}{m}$ scales as $\Omega^{\frac{4}{1+\eta}}$ after including $1/N$ corrections due to the gauge field fluctuations.

In the following we will directly compare our perturbative result for Π_{00} with the density-density correlation function in the Fermi liquid theory. Our goal is to find out whether the perturbative result can be consistent with a Fermi liquid theory made up of quasi-particles with a divergent effective mass m^* as suggested, for example, by Eq.(1.19). First we consider the limit $\Omega = 0, q \to 0$, where it is well known that the Fermi liquid theory predicts

$$\Pi_{00}(\mathbf{q} \to 0, \Omega = 0) = \frac{\Pi_{00}^*(\mathbf{q} \to 0, \Omega = 0)}{1 + f_{0s}\, \Pi_{00}^*(\mathbf{q} \to 0, \Omega = 0)} \,, \tag{2.69}$$

where $\Pi_{00}^* = -\int \frac{d^2p}{(2\pi)^2} \frac{n^0_{\mathbf{p}} - n^0_{\mathbf{p}-\mathbf{q}}}{\Omega - (\epsilon^*_{\mathbf{p}} - \epsilon^*_{\mathbf{p}-\mathbf{q}})}$ is the free fermion response fuunction with an effective mass m^* and f_{0s} is the angular average of the Fermi liquid interaction parameter $f_{\mathbf{p}\mathbf{p}'}$. In two dimensions, for small q limit,

$$\Pi_{00}^*(\mathbf{q}, \Omega) = -\frac{m^*}{2\pi} \left(1 - \frac{x}{\sqrt{x^2 - 1}}\, \theta(x^2 - 1) + i \frac{x}{\sqrt{1 - x^2}}\, \theta(1 - x^2) \right) \,, \tag{2.70}$$

where $x = \Omega / v_F^* q$. In Euclidean space, the above formula can be reduced to

$$\Pi_{00}^*(\mathbf{q}, i\nu) = -\frac{m^*}{2\pi}\left(1 - \frac{|\alpha|}{\sqrt{1+\alpha^2}}\right) , \tag{2.71}$$

where $\alpha = \nu / v_F^* q$. Since $\Pi_{00}^*(\mathbf{q} \to 0, \Omega = 0) \propto m^*$, the fact that $\Pi_{00}(\mathbf{q} \to 0, \Omega = 0)$ is not enhanced implies that f_{0s} is a finite constant. However, this does not imply that the leading order term in the perturbative expansion of f_{0s} is finite. In fact, it is clear from an expansion of Eq. (2.69) that if the leading order correction to m is singular, then the contribution to f_{0s} at the same order should be also singular since Π_{00} has no singular correction in the lowest order perturbation theory.

Next we consider the full q, Ω dependence of Π_{00} for small q and Ω. We are motivated by the belief that, in the Fermi liquid theory, Im $\Pi_{00}(\mathbf{q}, \Omega)$ should exhibit the edge of the particle-hole continuum along the line $\Omega = v_F^* q$. However, when $\Omega \neq 0$, a simple formula such as Eq. (2.69) does not exist for $\Pi_{00}(\mathbf{q}, \Omega)$. In particular, $\Pi_{00}(\mathbf{q}, \Omega)$ in general depends on the higher moment angular average of the Landau functions, and not just f_{0s}. Nevertheless, the Fermi liquid theory makes a precise prediction for $\Pi_{00}(\mathbf{q}, \Omega)$ for all q, Ω in terms of m^* and the interaction parameter $f_{\mathbf{pp}'}$. This is given by the quantum Boltzmann equation for the quasi-particle distribution function $n_{\mathbf{p}} = n_{\mathbf{p}}^0 + \delta n_{\mathbf{p}}$ in the Fermi liquid theory, where $n_{\mathbf{p}}^0$ is the distribution function for the free fermion system with an effective mass m^*:

$$\begin{aligned} &\left[\, \Omega - (\epsilon^*_{\mathbf{p}+\mathbf{q}/2} - \epsilon^*_{\mathbf{p}-\mathbf{q}/2}) \,\right] \delta n_{\mathbf{p}} \\ &-(n^0_{\mathbf{p}+\mathbf{q}/2} - n^0_{\mathbf{p}-\mathbf{q}/2}) \left[\, U(\mathbf{q}, \Omega) + \int \frac{d^2p'}{(2\pi)^2}\, f_{\mathbf{pp}'}\, \delta n_{\mathbf{p}'}(\mathbf{q}, \Omega) \,\right] = 0 \, . \end{aligned} \tag{2.72}$$

Here $\epsilon^*_{\mathbf{p}}$ is the quasi-particle energy, $U(\mathbf{q}, \Omega)$ is the external potential, and $f_{\mathbf{pp}'}$ is the Fermi-liquid interaction parameter. The linear response of $\delta n_{\mathbf{p}}$ to the external potential can be calculated from Eq. (2.69) (to the first order in $f_{\mathbf{pp}'}$):

$$\begin{aligned} \delta n_{\mathbf{p}}(\mathbf{q}, \Omega) &= \left[\, c_{\mathbf{p}} + \int \frac{d^2p'}{(2\pi)^2}\, c_{\mathbf{p}} f_{\mathbf{pp}'} c_{\mathbf{p}'} \,\right] U(\mathbf{q}, \Omega) \\ c_{\mathbf{p}} &= \frac{n^0_{\mathbf{p}+\mathbf{q}/2} - n^0_{\mathbf{p}-\mathbf{q}/2}}{\Omega - (\epsilon^*_{\mathbf{p}+\mathbf{q}/2} - \epsilon^*_{\mathbf{p}-\mathbf{q}/2})} \, . \end{aligned} \tag{2.73}$$

The change in the density of the fermions $\delta\rho(\mathbf{q}, \Omega) = \int \frac{d^2p}{(2\pi)^2}\, \delta n_{\mathbf{p}}(\mathbf{q}, \Omega)$ is given by

$$\begin{aligned} \frac{\delta\rho(\mathbf{q}, \Omega)}{U(\mathbf{q}, \Omega)} &= -\Pi_{00}(\mathbf{q}, \Omega) \\ &= \int \frac{d^2p}{(2\pi)^2} \frac{n^0_{\mathbf{p}} - n^0_{\mathbf{p}-\mathbf{q}}}{\Omega - (\epsilon^*_{\mathbf{p}} - \epsilon^*_{\mathbf{p}-\mathbf{q}})} + \int \frac{d^2p\, d^2p'}{(2\pi)^4}\, c_{\mathbf{p}} f_{\mathbf{pp}'} c_{\mathbf{p}'} + \cdots \, , \end{aligned} \tag{2.74}$$

where the ellipses represents the higher order terms in $f_{\mathbf{pp}'}$. The second term is just the diagram given in Figure 2-1 (e), but with a frequency-independent interaction $f_{\mathbf{pp}'}$.

Let us now examine what happens to the edge in the particle-hole continuum according to our perturbative results. The gauge interaction may induce non-zero Fermi-liquid interaction function $f_{\mathbf{pp}'}$ and a change in the Fermi velocity δv_F. From Eq. (2.71) and Eq. (2.74), a change in the Fermi velocity δv_F and the appearance of the Fermi liquid interaction parameter induce the following change in the density-density correlation function:

$$\delta\Pi_{00} = -\frac{\delta v_F}{v_F}\left(-\Pi^*_{00} + \frac{k_F}{2\pi v_F}\frac{|\alpha|}{(1+\alpha^2)^{3/2}}\right) - \int \frac{d^2p\, d^2p'}{(2\pi)^4}\, c_{\mathbf{p}} f_{\mathbf{pp}'} c_{\mathbf{p}'} \,. \tag{2.75}$$

If we assume a power law behavior for $f_{\mathbf{pp}'} \sim \frac{1}{|\mathbf{p}-\mathbf{p}'|^\lambda}$ with $\lambda < 1$ (*i.e.*, finite f_{0s}), one can show that the second term in Eq. (2.75) cannot produce the singular term $(1+\alpha^2)^{-3/2}$ near $\alpha^2 = -1$. To prove this argument, let us perform the integration over $|\mathbf{p}|$ and $|\mathbf{p}'|$ in the small q limit, yielding

$$\int \frac{d^2p\, d^2p'}{(2\pi)^4}\, c_{\mathbf{p}} f_{\mathbf{pp}'} c_{\mathbf{p}'} = \frac{4k_F^2}{(2\pi)^4}\int d\theta_{\mathbf{pq}}\, d\theta_{\mathbf{p}'\mathbf{q}}\, \frac{q^2 \cos\theta_{\mathbf{pq}} \cos\theta_{\mathbf{p}'\mathbf{q}}\, f_{\mathbf{pp}'}}{(\Omega - v_F q\cos\theta_{\mathbf{pq}})(\Omega - v_F q \cos\theta_{\mathbf{p}'\mathbf{q}})}\,, \tag{2.76}$$

where $\theta_{\mathbf{pq}}$ ($\theta_{\mathbf{p}'\mathbf{q}}$) is the angle between $\mathbf{p}$ and $\mathbf{q}$ ($\mathbf{p}'$ and $\mathbf{q}$). In order to obtain the leading singularity near $\Omega = v_F q$, the above expression can be further simplified:

$$\begin{aligned}&\int \frac{d^2p\, d^2p'}{(2\pi)^4}\, c_{\mathbf{p}} f_{\mathbf{pp}'} c_{\mathbf{p}'} \\ &= \frac{4k_F^2}{(2\pi)^4 v_F^2}\int d\theta_{\mathbf{pq}}\, d\theta_{\mathbf{p}'\mathbf{q}}\, \frac{f_{\mathbf{pp}'}}{\left[\left(\frac{\Omega}{v_F q}-1\right)+\frac{1}{2}\theta^2_{\mathbf{pq}}\right]\left[\left(\frac{\Omega}{v_F q}-1\right)+\frac{1}{2}\theta^2_{\mathbf{p}'\mathbf{q}}\right]}\,.\end{aligned} \tag{2.77}$$

For $f_{\mathbf{pp}'} \propto \frac{1}{|\theta_{\mathbf{pq}}-\theta_{\mathbf{p}'\mathbf{q}}|^\lambda}$ with $\lambda < 1$, the above integral can be estimated through a scaling argument. We find

$$\int \frac{d^2p\, d^2p'}{(2\pi)^4}\, c_{\mathbf{p}} f_{\mathbf{pp}'} c_{\mathbf{p}'} \propto \frac{1}{\left(\frac{\Omega}{v_F q}-1\right)^{\frac{2+\lambda}{2}}}\,, \tag{2.78}$$

which is less divergent than $(1+\alpha^2)^{-3/2}$ term that leads to $\left(\frac{\Omega}{v_F q}-1\right)^{-3/2}$ divergence. Thus there is no cancellation between the first and the second terms in Eq. (2.75). If δv_F diverges at small frequencies, we can conclude that $\delta\Pi_{00}$ will diverge in the limit $\nu \to 0$ with $\nu/v_F q$ fixed, which contradicts to our two-loop result from Eq. (2.64) that shows no such divergent term. Similar results also hold for the transverse current-current response function.

The argument above assumes a power law behavior for $f_{\mathbf{pp}'} \propto \frac{1}{|\theta_{\mathbf{pq}}-\theta_{\mathbf{p}'\mathbf{q}}|^\lambda}$. As $\lambda \to 1$, another possibility needs to be considered, namely $f_{\hat{\mathbf{p}}\hat{\mathbf{p}}'} \propto \delta(\hat{\mathbf{p}} - \hat{\mathbf{p}}')$. This satisfies the condition that f_{0s} is finite. From Eq. (2.77) it is clear that this will lead to a term of order $(1+\alpha^2)^{-3/2}$ which may cancel the first term in Eq. (2.75). However, in this case, we shall argue that, at least at zero temperature, $f_{\hat{\mathbf{p}}\hat{\mathbf{p}}'} = \zeta\, \delta(\hat{\mathbf{p}} - \hat{\mathbf{p}}')$ is equivalent to a shift in the Fermi velocity by $v_F \to v_F + \zeta k_F/(2\pi)^2$. At zero temperature the excitation can be described by a distortion of the Fermi surface in the direction $\hat{\mathbf{p}}$ by an amount $\delta\nu_{\hat{\mathbf{p}}} = \int d|\mathbf{p}|\ \delta n_{\mathbf{p}}$. The original Landau's expression of the free energy density takes the form:

$$\begin{aligned} \delta F &= \int \frac{d^2p}{(2\pi)^2}\ v_F(|\mathbf{p}| - k_F)\delta n_{\mathbf{p}} + \frac{1}{2}\int \frac{d^2p\ d^2p'}{(2\pi)^4}\ f_{\mathbf{p}\mathbf{p}'}\delta n_{\mathbf{p}}\delta n_{\mathbf{p}'} \\ &= \int \frac{k_F\ d\hat{\mathbf{p}}}{(2\pi)^2}\ \frac{1}{2} v_F(\delta\nu_{\hat{\mathbf{p}}})^2 + \frac{1}{2}\int \frac{k_F^2\ d\hat{\mathbf{p}}\ d\hat{\mathbf{p}}'}{(2\pi)^4}\ f_{\hat{\mathbf{p}}\hat{\mathbf{p}}'}\delta\nu_{\hat{\mathbf{p}}}\delta\nu_{\hat{\mathbf{p}}'}\ . \end{aligned} \tag{2.79}$$

It is then clear that $f_{\hat{\mathbf{p}}\hat{\mathbf{p}}'} = \zeta\ \delta(\hat{\mathbf{p}} - \hat{\mathbf{p}}')$ is equivalent to $v_F \to v_F + \zeta k_F/(2\pi)^2$. The same result can be also obtained by performing an integral over $|\mathbf{p}|$ in Eq. (2.72), which leads to

$$(\Omega - v_F q\ \cos\theta)\ \delta\nu_{\hat{\mathbf{p}}} - q\ \cos\theta \left[U(\mathbf{q},\Omega) + \int \frac{k_F\ d\hat{\mathbf{p}}'}{(2\pi)^2}\ f_{\hat{\mathbf{p}}\hat{\mathbf{p}}'}\ \delta\nu_{\hat{\mathbf{p}}'} \right] = 0 \tag{2.80}$$

in the small q limit. Thus we see that, at zero temperature, all response functions to an external perturbation can be described by a Landau theory with a non-divergent effective mass in the small q limit. However, it is also possible that the same response function can be described by a Landau-Fermi-liquid theory of which both effective mass and $f_{\mathbf{p}\mathbf{p}'}$ have divergent perturbative corrections.

An examination of Eq. (2.63) shows that after analytic continuation, the factor $(1+\alpha^2)^{-5/2}$ diverges at $\Omega = v_F q$, even though its coefficient vanishes for $\Omega \to 0$. In the following we attempt an interpretation of the result. We can write our perturbative result Eq. (2.63) as, near $\Omega = v_F q$,

$$\text{Im}\ \Pi_{00}(\mathbf{q},\Omega) = \text{Im}\ \Pi^0_{00}(\mathbf{q},\Omega) + \alpha_0 \frac{\partial\ \text{Im}\ \Pi^0_{00}(\mathbf{q},\Omega)}{\partial\Omega} + \gamma_0 \frac{\partial^2\ \text{Im}\ \Pi^0_{00}(\mathbf{q},\Omega)}{\partial\Omega^2}\ , \tag{2.81}$$

where Π^0_{00} is given by Eq. (2.70) with $m^* \to m$, and

$$\begin{aligned} \alpha_0 &= \frac{a_2}{k_F^{\eta-2}\chi} q\ , \\ \gamma_0 &= \frac{1+\eta}{8\pi^2(5+\eta)} \frac{1}{\cos\left(\frac{2\pi}{1+\eta}\right)} \frac{1}{k_F\gamma} \frac{1}{v_F q} \left(\frac{\gamma\Omega}{\chi}\right)^{\frac{4}{1+\eta}} q^2\ , \end{aligned} \tag{2.82}$$

where a_2 is a constant. The existence of $\partial\ \text{Im}\ \Pi^0_{00}(\mathbf{q},\Omega)/\partial\Omega$ term in Eq. (2.81) signifies that there is a finite non-singular (see α_0 in Eq. (2.82)) shift in v_F, which also arises in the usual Fermi liquid theory. To interpret the second derivative term, we note that Eq. (2.81) is consistent with (apart from the term proportional to α_0)

$$\text{Im}\ \Pi_{00}(\mathbf{q},\Omega) = \frac{1}{2}\left[\ \text{Im}\ \Pi^0_{00}(\mathbf{q},\Omega+\Gamma) + \text{Im}\ \Pi^0_{00}(\mathbf{q},\Omega-\Gamma)\ \right] \tag{2.83}$$

if $\Gamma = \sqrt{2\gamma_0}$. We recall that $\text{Im}\ \Pi^0_{00}(\mathbf{q},\Omega)$ has a discontinuity at $\Omega = v_F q$, corresponding to the edge of the particle-hole continuum. Eq. (2.83) has the natural interpretation of a smearing of the discontinuity at a shifted (due to a shift in v_F)

edge of the particle-hole continuum by the amount Γ. Setting $v_F q \propto \Omega$, we find that

$$\Gamma \propto \Omega^{1+\frac{3-\eta}{2+2\eta}} . \tag{2.84}$$

Note that for $\eta < 3$, $\Gamma < \Omega$ so that the above picture is a self-consistent one. We also note that Γ is proportional to the square root of the coupling constant or $1/N$, and is therefore nonanalytic. We are not certain if any further physical meaning can be ascribed to the energy scale Γ.

2.6 Conclusion

In this chapter, we studied properties of gauge-invariant correlation functions in a two-dimensional fermion system coupled to a gauge field. We find the physical picture emerged from those gauge-invariant correlation functions to be very different from those obtained from gauge-dependent one-particle Green's function. The corrections to the Fermi-liquid two-particle correlation functions are found to be non-divergent and sub-leading to the Fermi-liquid contributions up to two-loop order, and there is no need to go beyond the perturbation theory at this order.

However, it is still possible that singular corrections to the gauge-invariant two-particle correlation functions may appear in some special cases, such as $q = 2k_F$. In fact, Altshuler, Ioffe, and Millis [17] found that the density-density correlation function at $q = 2k_F$ shows indeed singular behaviors. Also, since we do not have quasi-particles to serve as the underpinning of the Fermi-liquid-like behavior for Π_{00} and Π_{11}, it is possible that singularity shows up in some other response functions. Nevertheless, the perturbative result should serve as a test for any theory such as renormalization group analysis [24] which attempts to go beyond perturbation theory.

Chapter 3

Compressibility and the Energy Gap of Composite Fermions near $\nu = 1/2$

3.1 Introduction

As discussed in chapter 1, the gauge field fluctuations give rise to a singular contribution to the self-energy in the one-particle Green's function of the composite fermions [6, 11]. The singular self-energy correction leads to a divergent effective mass of the composite fermions at $\nu = 1/2$ [6]. If one applied this divergent effective mass to the Shubnikov-de Haas effect near $\nu = 1/2$, one would find that the gap Δ_p of $\nu = p/(2p \pm 1)$ FQH state goes down faster than $1/p$ as $p \to \infty$:

$$p \, \Delta_p \to 0 \qquad \text{as} \qquad p \to \infty \, . \tag{3.1}$$

However, the one-particle Green's function of the composite fermions is not gauge invariant. Therefore, it is not clear whether the divergent effective mass in the one-particle Green's function is related to the above energy gap Δ_p which is measurable in real experiments.

In chapter 2, we have examined several gauge invariant two-particle correlation functions for all ratios of ω and q [13]. We found that, at low energies and in the long-wavelength limit, the gauge field fluctuations do not cause any divergent correction (up to two-loop level), and the two-particle correlation functions have the Fermi-liquid forms with a finite effective mass if one assumes a non-singular Fermi-liquid-parameter-function $f_{\mathbf{pp}'}$ [13]. Fermi liquid form of the density-density correlation function in the small q and ω limit was also found in the eikonal approximation [19] even though the result is not the same as that of the two-loop perturbative calculation [13]. Altshuler, Ioffe, and Millis also examined the two-particle correlation functions and especially found peculiar behaviors near $q = 2k_F$ [14].

We would like to mention that Fermi-liquid theory with a finite effective mass is not the conclusive interpretation of the behaviors of the density-density correlation function in the long wavelength and the low frequency limit. That is, it is still possible that the effect of the divergent effective mass may be cancelled by a contribution from a singular Fermi-liquid-parameter-function $f_{\mathbf{pp}'}$ so that the density-density correlation function for the long wavelength and the low energy limit behaves as if the effective mass is finite [16]. Indeed, Stern and Halperin [16] calculated the energy gap of the system from the one-particle Green's function of the composite fermions in a finite effective magnetic field ΔB. They argued that even though the one-particle Green's function is not gauge-invariant, the edge of the spectral function at zero temperature, across which the spectral function vanishes, should be gauge-invariant. By identifying the region where the spectral function vanishes, they found an energy gap which is in agreement with the previous self-consisteny treatment [6]. In view of the complexity of the problem, we feel that it is important to investigate whether the effect of the large enhancement of the effective mass will show up in some gauge-invariant response functions. In this paper, we calculate the lowest order correction (due to the gauge field) to the finite temperature compressibility as a function of an effective cyclotron frequency $\Delta\omega_c = \frac{e\Delta B}{mc}$ (where m is the bare mass of the fermions) in the limit of large p, *i.e.*, near $\nu = 1/2$. We find that when a chemical potential μ lies exactly at the middle of the successive effective Landau levels, for $T \ll \Delta\omega_c$, the compressibility behaves as

$$\frac{\partial n}{\partial \mu} \propto e^{-\Delta\omega_c/2T}\left(1 + \frac{A(\eta)}{\eta - 1}\frac{(\Delta\omega_c)^{\frac{2}{1+\eta}}}{T}\right) , \tag{3.2}$$

where $A(\eta)$ is a η-dependent positive dimensionful constant. Here, we assume that the interaction between the fermions has the form: $v(\mathbf{q}) = V_0/q^{2-\eta}(1 \le \eta \le 2)$. If we interpret the activation energy as a renormalized energy gap $\Delta\omega_c^*$, *i.e.*, $\frac{\partial n}{\partial \mu} \propto e^{-\Delta\omega_c^*/2T}$, it is given by $\Delta\omega_c^* \approx \Delta\omega_c\left(1 - \frac{2A(\eta)}{\eta-1}(\Delta\omega_c)^{-\frac{\eta-1}{\eta+1}}\right)$. If we write $\Delta\omega_c^* = \frac{e\Delta B}{m^* c}$, the above result is consistent with a divergent correction to the effective mass $m^*/m \approx 1 + \frac{2A(\eta)}{\eta-1}(\Delta\omega_c)^{-\frac{\eta-1}{\eta+1}}$ because A should be proportional to a small expansion parameter, which is $1/N$ for a large N generalized model. In particular, for the Coulomb interaction ($\eta = 1$), $m^*/m \approx 1 + 2A(\eta = 1)\ \ln\ (\epsilon_F/\Delta\omega_c)$ (ϵ_F is the Fermi energy) as predicted in terms of a self-consistent argument [6, 16].

We would like to remark that a comparision with the recent experimental measurements [2, 3] of the energy gap is complicated by the large impurity effects. The disordered potential due to the impurities causes a spatial fluctuation of the fermion density distribution, which is equivalent to a large spatial fluctuation of the Chern-Simons magnetic flux or ΔB. This means that there is a range of ΔB controlled by the degree of disorder around the filling factor $\nu = 1/2$, where impurity effects are very important. In reality, this is the region where the gap measurement is not possible due to the suppression of the amplitude of the Shubnikov-de Haas effect. We feel that a deeper understanding of the impurity effects is necessary before a recent experimental report [3] of an increase in the effective mass near the boundary of the disorder dominated region can be properly interpreted.

Before the main discussion, we would like to point out that there is a gauge-invariant (for the Chern-Simons gauge field) one-particle Green's function — the Green's function of the physical electrons, which does not have a Fermi-liquid form

[25] even though the two-particle Green's functions are similar to those of the Fermi liquid with a finite or divergent effective mass. In the first place, the electrons see a strong magnetic field and the electron Green's function does not have any singularity at k_F. Secondly the spectral weight of the electron Green's function is exponentially small at low energies even for the Coulomb interaction, which is very different from the Fermi liquid result [25]. Thus the $\nu = 1/2$ state really represents a new class of metallic state.

The remainder of this chapter is organized as follows. In section 3.2, we describe a method to calculate the lowest-order correction to the compressibility $\frac{\partial n}{\partial \mu}$, where n is the density of the composite fermions. In section 3.3, the compressibility of the fermions is calculated for $T \ll \Delta\omega_c \ll \mu$ when the chemical potential μ lies exactly at the middle of the two successive effective Landau levels. In section 3.4, we discuss and contrast two different methods of evaluating the compressibility and emphasize the gauge-invariant nature of the method used in this paper. We discuss and interpret our results in section 3.5.

3.2 The Model and the Compressibility

In the presence of finite ΔB ($\nu \neq 1/2$), after integrating out the fermions and including gauge field fluctuations within the random phase approximation, the effective action of the gauge field can be obtained as [6]

$$S_{\text{eff}} = \frac{1}{2} \int \frac{d^2 q}{(2\pi)^2} \frac{d\omega}{2\pi} \; \delta a^*_\mu(\mathbf{q}, \omega) \; D^{-1}_{\mu\nu}(\mathbf{q}, \omega, \Delta\omega_c) \; \delta a_\nu(\mathbf{q}, \omega) \; , \tag{3.3}$$

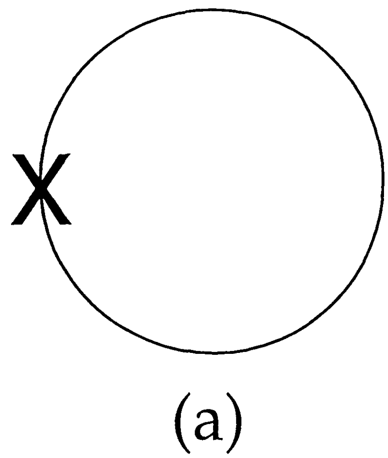
where $D^{-1}_{\mu\nu}(\mathbf{q}, \omega, \Delta\omega_c)$ was calculated by several authors [6, 5, 36, 37]. For our purpose, the 2×2 matrix form for $D^{-1}_{\mu\nu}$ is sufficient so that $\mu, \nu = 0, 1$ and 1 represents the direction that is perpendicular to $\mathbf{q}$ [6].

The compressibility of the fermions $\frac{\partial n}{\partial \mu}(\mu, \Delta\omega_c)$ as a function of chemical potential μ and an effective cyclotron frequency $\Delta\omega_c = \frac{e\Delta B}{mc}$ can be obtained from $n(\mu, \Delta\omega_c) = -\frac{\partial \Omega}{\partial \mu}$ (n is the density of the fermions), *i.e.*, $\frac{\partial n}{\partial \mu} = -\frac{\partial^2 \Omega}{\partial \mu^2}$. The density of the free fermions $n_0(\mu, \Delta\omega_c)$ and the lowest order correction $n_1(\mu, \Delta\omega_c)$ due to the transverse part of the gauge field fluctuations are given by the diagrams in Figure 3-1 (a) and (b) respectively. These contributions can be obtained from the relations $n_0(\mu, \Delta\omega_c) = -\frac{\partial \Omega_0}{\partial \mu}$ and $n_1(\mu, \Delta\omega_c) = -\frac{\partial \Omega_1}{\partial \mu}$, where Ω_0 and Ω_1 are the thermodynamic potential of the free fermions and the lowest order correction to the thermodynamic potential given by the diagrams in Figure 3-2 (a) and (b) respectively.

The density of the free fermions $n_0(\mu, \Delta\omega_c)$ at finite temperatures can be written as

$$n_0(\mu, \Delta\omega_c) = \frac{m(\Delta\omega_c)}{2\pi} \sum_l n_F(\xi_l) \; , \tag{3.4}$$

where $\xi_l = (l + 1/2)(\Delta\omega_c) - \mu$ and $n_F(x) = \frac{1}{e^{x/T}+1}$. Thus the compressibility of the



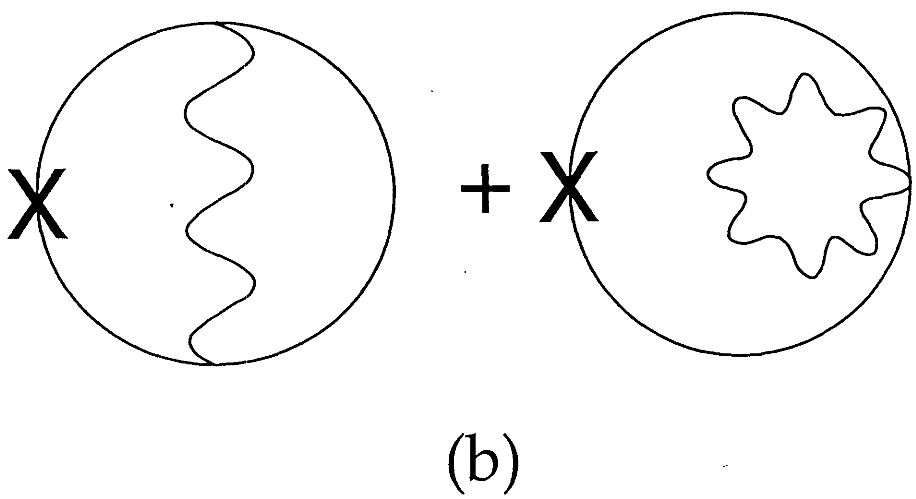


Figure 3-1: (a) The diagram that represents the density of the free fermions in an effective magnetic field ΔB. (b) The lowest order correction to the density of the fermions due to the gauge field fluctuation. Here the solid line represents the bare electron propagator. The wavy line denotes the RPA gauge field propagator which is given by the diagram in Figure 3-2 (a).

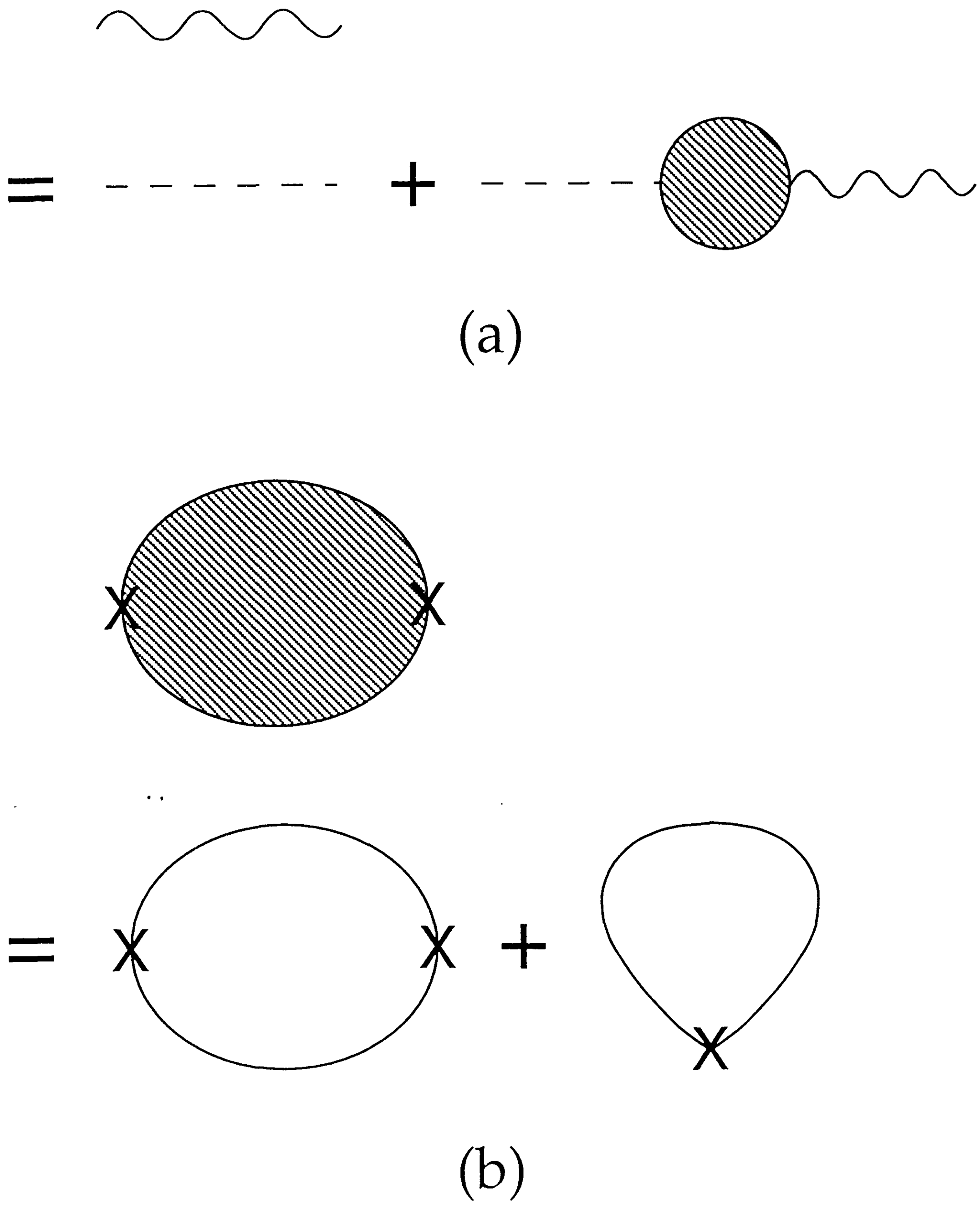


Figure 3-2: (a) The wavy line denotes the RPA gauge field propagator and the dashed line is the bare gauge field propagator. Here the hatched bubble (b) represents the transverse part of the polarization bubble.

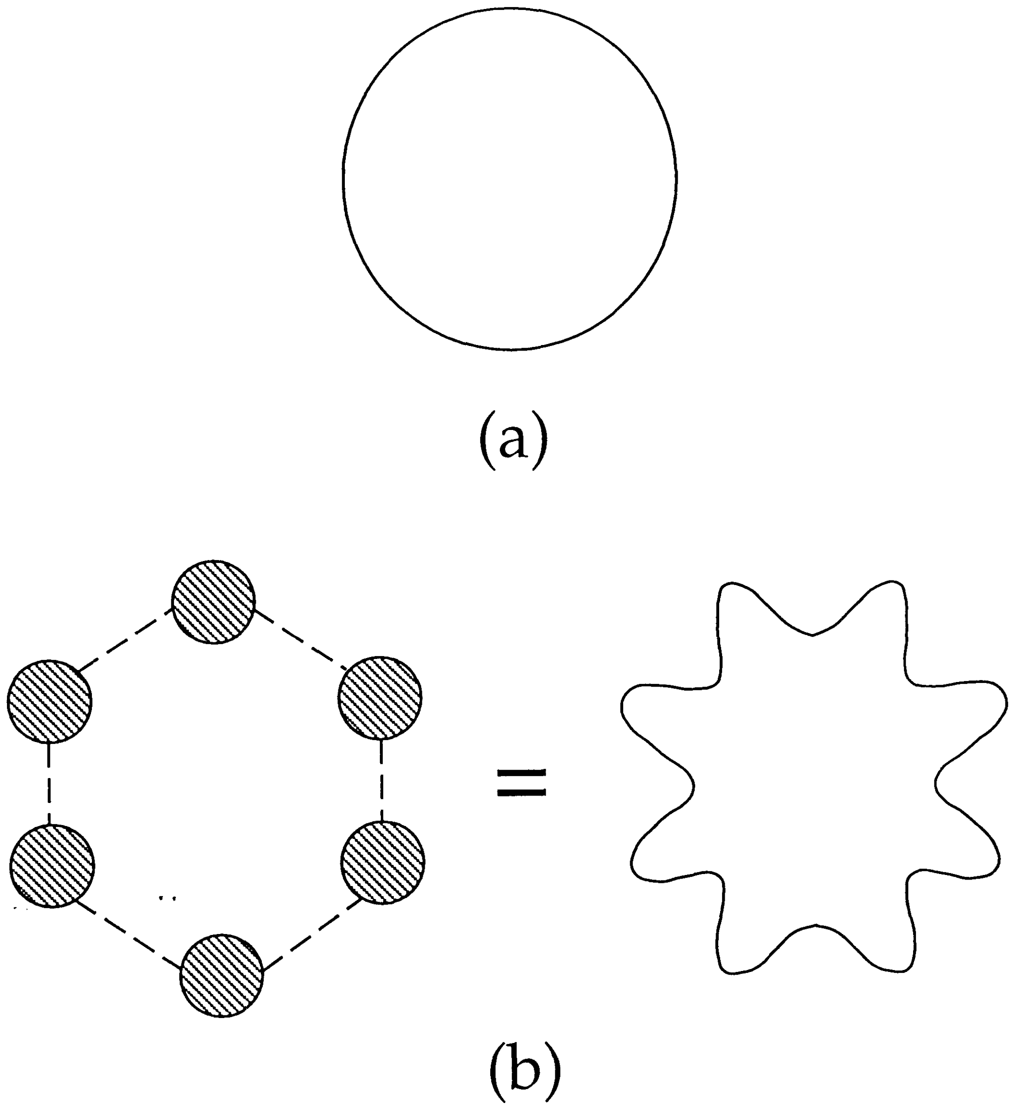


Figure 3-3: The diagrams that correspond to the thermodynamic potential of the free fermions (a) and the gauge field contribution (b) to the thermodynamic potential.

free fermions is given by

$$\frac{\partial n_0}{\partial \mu} = \frac{m}{2\pi} \frac{\Delta\omega_c}{T} \sum_l n_F(\xi_l)(1 - n_F(\xi_l)) \ . \tag{3.5}$$

The lowest order correction (due to the transverse part of the gauge field) to the density of the fermions can be obtained from

$$n_1(\mu, \Delta\omega_c) = T \sum_{i\nu_n} \sum_{\mathbf{q}} D_{11}(\mathbf{q}, i\nu_n) \ \frac{\partial}{\partial\mu} \Pi_{11}(\mathbf{q}, i\nu_n) \ , \tag{3.6}$$

where $\nu_n = 2\pi n T$ is the Matsubara frequency. Here Π_{11} is the transverse part of the fermion polarization bubble:

$$\Pi_{11}(\mathbf{q}, i\nu_n) = -\sum_{lm} |M_{lm}(\mathbf{q})|^2 \ \frac{n_F(\xi_l) - n_F(\xi_m)}{i\nu_n - \xi_m + \xi_l} - \frac{1}{m}\left(\frac{m\Delta\omega_c}{2\pi} \sum_l n_F(\xi_l)\right) \ , \tag{3.7}$$

where $|M_{lm}(\mathbf{q})|^2$ comes from the form of the current-current vertex and is calculated by several authors [5, 36, 37]. After analytic continuation $i\nu_n \to \nu + i0^+$, one gets the real part and the imaginary part of the retarded polarization function:

$$\begin{aligned}
\Pi'_{11}(\mathbf{q}, \nu) &= -\sum_{lm} |M_{lm}(\mathbf{q})|^2 \ \frac{n_F(\xi_l) - n_F(\xi_m)}{\nu - \xi_m + \xi_l} - \frac{1}{m}\left(\frac{m\Delta\omega_c}{2\pi} \sum_l n_F(\xi_l)\right) \ , \\
\Pi''_{11}(\mathbf{q}, \nu) &= \pi \sum_{lm} |M_{lm}(\mathbf{q})|^2 \ [\ n_F(\xi_l) - n_F(\xi_m)\ ] \ \delta(\nu - \xi_m + \xi_l) \ .
\end{aligned} \tag{3.8}$$

Here we use the convention that A' and A'' represent the real and the imaginary parts of a quantity A. Now the correction to the compressibility can be obtained as

$$\frac{\partial n_1}{\partial \mu} = T \sum_{i\nu_n} \sum_{\mathbf{q}} \left[\ D_{11}(\mathbf{q}, i\nu_n) \ \frac{\partial^2}{\partial \mu^2} \Pi_{11}(\mathbf{q}, i\nu_n) \ + \ \frac{\partial}{\partial\mu} D_{11}(\mathbf{q}, i\nu_n) \ \frac{\partial}{\partial\mu} \Pi_{11}(\mathbf{q}, i\nu_n) \ \right] \ . \tag{3.9}$$

For calculational convenience, we introduce $\widetilde{D}_{11}(\mathbf{q}, i\nu_n)$ which does not depend on μ. Then the correction to the physical fermion density $n_1(\mu, \Delta\omega_c)$ can be obtained from $n_1(\mu, \Delta\omega_c) = -\frac{\partial \Omega_{\text{toy}}}{\partial \mu}$, where

$$\Omega_{\text{toy}} = T \sum_{i\nu_n} \sum_{\mathbf{q}} \widetilde{D}_{11}(\mathbf{q}, i\nu_n) \ \Pi_{11}(\mathbf{q}, i\nu_n) \ , \tag{3.10}$$

and replace $\widetilde{D}_{11}(\mathbf{q}, i\nu_n)$ by $D_{11}(\mathbf{q}, i\nu_n)$ after taking the derivative with respect to μ. Using the spectral representation, one can write Ω_{toy} as

$$\begin{aligned}
\Omega_{\text{toy}} &= \Omega_a + \Omega_b \ , \\
\Omega_a &= \sum_{\mathbf{q}} \int_{-\infty}^{\infty} \frac{dx}{\pi} \ n_B(x) \ \widetilde{D}'_{11}(\mathbf{q}, x) \ \Pi''_{11}(\mathbf{q}, x) \ ,
\end{aligned}$$

$$\Omega_b = \sum_{\mathbf{q}} \int_{-\infty}^{\infty} \frac{dx}{\pi} \; n_B(x) \; \widetilde{D}''_{11}(\mathbf{q},x) \; \Pi'_{11}(\mathbf{q},x) \; , \tag{3.11}$$

where $n_B(x) = \frac{1}{e^{x/T}-1}$. After taking the derivative with respect to μ and replacing $\widetilde{D}_{11}$ by D_{11}, we get the lowest order correction to the density of the fermions:

$$\begin{aligned}
n_1 &= n_a + n_b \; , \\
n_a &= -\sum_{\mathbf{q}} \int_{-\infty}^{\infty} \frac{dx}{\pi} \; n_B(x) \; D'_{11}(\mathbf{q},x) \; \frac{\partial}{\partial\mu} \Pi''_{11}(\mathbf{q},x) \; , \\
n_b &= -\sum_{\mathbf{q}} \int_{-\infty}^{\infty} \frac{dx}{\pi} \; n_B(x) \; D''_{11}(\mathbf{q},x) \; \frac{\partial}{\partial\mu} \Pi'_{11}(\mathbf{q},x) \; .
\end{aligned} \tag{3.12}$$

For the lowest order correction $\frac{\partial n_1}{\partial\mu}$ to the compressibility, the derivative with respect to μ should be taken for both D_{11} and Π_{11}. Thus $\frac{\partial n_1}{\partial\mu}$ can be written as

$$\begin{aligned}
\frac{\partial n_1}{\partial\mu} &= \frac{\partial n_a}{\partial\mu} + \frac{\partial n_b}{\partial\mu} \; , \\
\frac{\partial n_a}{\partial\mu} &= -\sum_{\mathbf{q}} \int_{-\infty}^{\infty} \frac{dx}{\pi} \; n_B(x) \left[D'_{11} \frac{\partial^2 \Pi''_{11}}{\partial\mu^2} + \frac{\partial D'_{11}}{\partial\mu} \frac{\partial \Pi''_{11}}{\partial\mu} \right] , \\
\frac{\partial n_b}{\partial\mu} &= -\sum_{\mathbf{q}} \int_{-\infty}^{\infty} \frac{dx}{\pi} \; n_B(x) \left[D''_{11} \frac{\partial^2 \Pi'_{11}}{\partial\mu^2} + \frac{\partial D''_{11}}{\partial\mu} \frac{\partial \Pi'_{11}}{\partial\mu} \right] .
\end{aligned} \tag{3.13}$$

Note that Eq. (3.13) is equivalent to Eq. (3.9). This procedure generates the diagrams for the compressibility, which are shown in Figure 3-4. In the next section, we evaluate the expressions for the compressibility.

3.3 The Finite Temperature Compressibility for $T \ll \Delta\omega_c \ll \mu$

In this section, we calculate the compressibility of the fermions as a function of $\Delta\omega_c$ and T in the limit $T \ll \Delta\omega_c$. First we would like to give a general discussion of the interaction effects on the compressibility. For free fermions at zero temperature and finite magnetic field, $\frac{dn}{d\mu} = \sum_m \delta(\mu - (n+\frac{1}{2}))\Delta\omega_c$ is the density of states. Each δ-function corresponds to a degenerate effective Landau level. The interaction has two kinds of effects on the compressibility $\frac{dn}{d\mu}$. First, the interaction effects split the degeneracy of the states in each effective Landau level (when the effective Landau level is partially filled). This effect spreads the δ-function in the free fermion compressibility into broadened peaks. The width of the peak (defined as the width of the region where $\frac{dn}{d\mu} \neq 0$) can be viewed as the width of the effective Landau bands (*i.e.*, the broadened effective Landau levels). Second, the interaction effects may shift the center of the effective Landau bands. However, since the average compressibility over many effective Landau levels is not changed by the transverse gauge field interaction, we expect that such an interaction can only cause a uniform shift of the center of

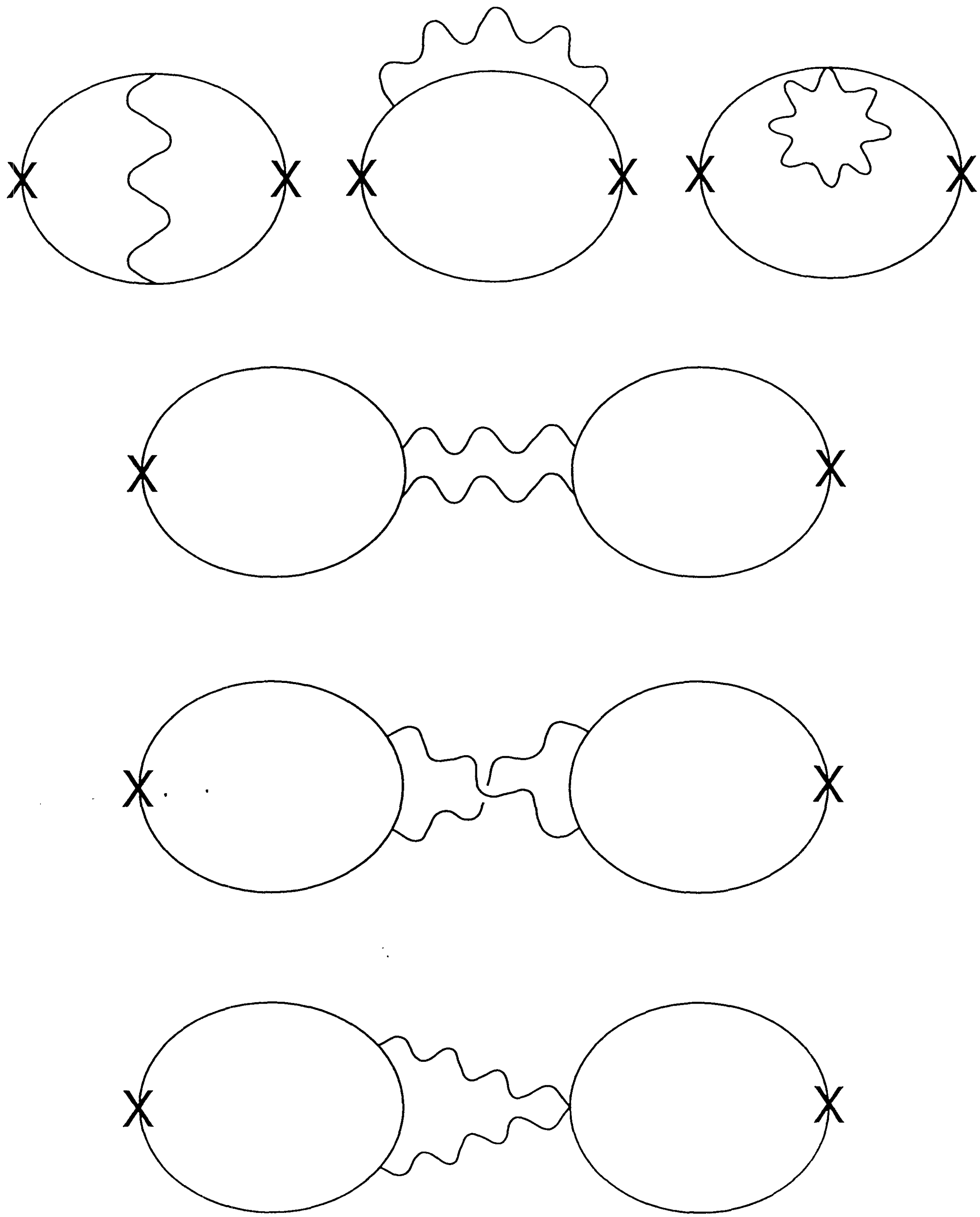


Figure 3-4: The diagrams that represent the lowest order corrections to the compressibility of the fermions.

the effective Landau bands, as one can see later in our explicit calculations. The activation energy gap measured in the transport experiments is given by the gap between the effective Landau bands. Thus the uniform shift is not important for the calculation of of the experimentally measurable activation energy gap. In the following calculations, we will assume that the chemical potential μ lies exactly at the middle of the two successive effective Landau levels, and investigate the activated behavior of the compressibility. In this case, the uniform shift of the center of the effective Landau bands is cancelled out and does not appear in the compressibility.

Let p be the number of filled effective Landau levels. For the free fermions, when $T \ll \Delta\omega_c$, we can expect that the compressibility shows a thermally activated behavior. In fact, from Eq. (3.5) and for $T \ll \Delta\omega_c$, it can be shown that at finite temperatures the compressibility of the free fermions can be written as

$$\frac{\partial n_0}{\partial \mu} = \frac{m}{2\pi}\,\frac{\Delta\omega_c}{T}\left(e^{-|\xi_p|/T} + e^{-\xi_{p+1}/T}\right) \;+\; \mathcal{O}(e^{-2|\xi_p|/T})\,. \tag{3.14}$$

Note that it becomes

$$\frac{\partial n_0}{\partial \mu} = \frac{m\Delta\omega_c}{\pi T}\; e^{-\Delta\omega_c/2T} \;+\; \mathcal{O}(e^{-\Delta\omega_c/T}) \tag{3.15}$$

for a chemical potential lying exactly at the middle of the Landau levels labeled by p and $p+1$. Our aim is to calculate the lowest order correction (due to the gauge field fluctuations) to the above free fermion result.

In order to calculate the lowest order correction $\frac{\partial n_1}{\partial \mu}$, we consider first $\Omega_{\rm toy} = \Omega_a + \Omega_b$. Substituting Eq. (3.8) to Eq. (3.11), we get

$$\begin{aligned}
\Omega_a &= \Omega_{a1} + \Omega_{a2}\,, \\
\Omega_{a1} &= \sum_{\mathbf{q}}\sum_{l} |M_{ll}(\mathbf{q})|^2\; \widetilde{D}'_{11}(\mathbf{q},0)\; n_F(\xi_l)(1-n_F(\xi_l))\,, \\
\Omega_{a2} &= \sum_{\mathbf{q}}\sum_{l\neq m} |M_{lm}(\mathbf{q})|^2\; \widetilde{D}'_{11}(\mathbf{q},\xi_m-\xi_l)\; n_F(\xi_m)(1-n_F(\xi_l))\,,
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
\Omega_b &= \Omega_{b1} + \Omega_{b2}\,, \\
\Omega_{b1} &= \sum_{\mathbf{q}}\int_0^\infty \frac{dx}{\pi}\,(1+2n_B(x))\;\widetilde{D}''_{11}(\mathbf{q},x)\left[-\sum_{lm}|M_{lm}(\mathbf{q})|^2\,\frac{n_F(\xi_l)-n_F(\xi_m)}{x-\xi_m+\xi_l}\right]\,, \\
\Omega_{b2} &= \sum_{\mathbf{q}}\int_0^\infty \frac{dx}{\pi}\,(1+2n_B(x))\;\widetilde{D}''_{11}(\mathbf{q},x)\left[-\frac{\Delta\omega_c}{2\pi}\sum_{l} n_F(\xi_l)\right]\,.
\end{aligned} \tag{3.17}$$

Now some explanations for each contribution are in order. Ω_{a1} and Ω_{a2} are contributions from the exchange interaction via the gauge field and represent the effect of the intra-Landau level and the inter-Landau level particle-hole excitations respectively. Ω_{b1} and Ω_{b2} are due to the thermal and the quantum (represented by $n_B(x)$ and 1 in the factor $1+2n_B(x)$) fluctuations of the gauge field. Note that the quantum contribution survives in the $T \to 0$ limit. In particular, Ω_{b2} comes from the diamagnetic coupling between the fermions and the gauge field. We also note that the intra-Landau level terms (with $l = m$) are associated with the splitting of the degenerate states in each Landau-level, and contribute to the spreading of the Landau-levels. On the other hand, the inter-Landau level terms (with $l \neq m$) will contribute to the shift of the center of the Landau bands.

The corresponding contributions to the density of the fermions are defined as $n_a = -\frac{\partial \Omega_a}{\partial \mu}$ and $n_b = -\frac{\partial \Omega_b}{\partial \mu}$. Thus the correction to the density of the fermions $\frac{\partial n_1}{\partial \mu}$ is given by $\frac{\partial n_1}{\partial \mu} = \frac{\partial n_a}{\partial \mu} + \frac{\partial n_b}{\partial \mu}$. Now we are going to find the contributions which are order of $e^{-|\xi_p|/T}$ or $e^{-\xi_{p+1}/T}$. Note that $\frac{\partial n_a}{\partial \mu} = \frac{\partial n_{a1}}{\partial \mu} + \frac{\partial n_{a2}}{\partial \mu}$, where $n_{a1} = -\frac{\partial \Omega_{a1}}{\partial \mu}$ and $n_{a2} = -\frac{\partial \Omega_{a2}}{\partial \mu}$. In the appendix A, we show that $\frac{\partial n_{a2}}{\partial \mu}$ is order of $e^{-2|\xi_p|/T}$ which is exponentially smaller than $e^{-|\xi_p|/T}$ or $e^{-\xi_{p+1}/T}$. It is also shown that $\frac{\partial n_b}{\partial \mu}$ is order of $e^{-2|\xi_p|/T}$ after a partial cancellation between Ω_{b1} and Ω_{b2} by the f-sum rule.

Now let us look at $\frac{\partial n_{a1}}{\partial \mu}$ for which a detailed expression is given in the appendix A. As mentioned before, we assume that we are very close to the half-filled state, *i.e.*, $\mu/\Delta\omega_c \gg 1$, which also corresponds to the large p limit. In this case, it can be shown that

$$\begin{aligned} \frac{\partial n_{a1}}{\partial \mu} &\approx -\frac{1}{T^2} \sum_{\mathbf{q}} \left[e^{-\xi_{p+1}/T} \, |M_{p+1p+1}(\mathbf{q})|^2 \, D'_{11}(\mathbf{q},0) \; + \; e^{-|\xi_p|/T} \, |M_{pp}(\mathbf{q})|^2 \, D'_{11}(\mathbf{q},0) \right] \\ &\quad + \mathcal{O}(e^{-2|\xi_p|/T}) \\ &\approx -\frac{1}{T^2} \left[e^{-\xi_{p+1}/T} + e^{-|\xi_p|/T} \right] \sum_{\mathbf{q}} |M_{pp}(\mathbf{q})|^2 \, D'_{11}(\mathbf{q},0) \; + \; \mathcal{O}(e^{-2|\xi_p|/T}) \; . \end{aligned} \tag{3.18}$$

For $\xi_{p+1} = |\xi_p| = \Delta\omega_c/2$, we get

$$\frac{\partial n_{a1}}{\partial \mu} \approx -\frac{2}{T^2} \, e^{-\Delta\omega_c/2T} \sum_{\mathbf{q}} |M_{pp}(\mathbf{q})|^2 \, D'_{11}(\mathbf{q},0) \; + \; \mathcal{O}(e^{-\Delta\omega_c/T}) \; . \tag{3.19}$$

Thus $\frac{\partial n_1}{\partial \mu} = \frac{\partial n_{a1}}{\partial \mu} \; + \; \mathcal{O}(e^{-\Delta\omega_c/T})$.

Now let us evaluate the following quantity.

$$I = \sum_{\mathbf{q}} |M_{pp}(\mathbf{q})|^2 \, D'_{11}(\mathbf{q},0) = 2 \sum_{\mathbf{q}} |M_{pp}(\mathbf{q})|^2 \int_0^\infty \frac{dy}{\pi} \, \frac{D''_{11}(\mathbf{q},y)}{y} \; . \tag{3.20}$$

Note that the matrix element $|M_{pp}(\mathbf{q})|^2$ comes from the vertex of the paramagnetic part of the current-current correlation function. For the large p limit or $\mu/\Delta\omega_c \gg 1$, we may use a semiclassical approximation $j \approx v_F \rho$, where j and ρ are the current and the density of the fermions. Thus $|M_{pp}(\mathbf{q})|^2$ can be approximated as $|M_{pp}(\mathbf{q})|^2 \approx v_F^2 |M_{pp}^{00}(\mathbf{q})|^2$, where $|M_{pp}^{00}(\mathbf{q})|^2$ is the corresponding matrix element for the density-density correlation function [5, 36, 37]. Using the above approximation, we get

$$|M_{pp}(\mathbf{q})|^2 \approx \frac{v_F^2}{2\pi l_c^2} \, e^{-X} \, \left[L_p^0(X) \right]^2 \; , \tag{3.21}$$

where $l_c^2 \equiv \frac{\hbar c}{e\Delta B}$, $X \equiv \frac{1}{2} q^2 l_c^2$, and $L_p^0(X)$ is a Laguerre polynomial. For the large p

limit, $L_p^\alpha(X)$ can be approximated as [38]

$$L_p^\alpha(X) \approx \frac{1}{\pi}\, e^{X/2}\, X^{-\frac{\alpha}{2}-\frac{1}{4}}\, p^{\frac{\alpha}{2}-\frac{1}{4}} \cos\left(2\sqrt{pX} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) . \tag{3.22}$$

We use $p \approx \mu/\Delta\omega_c$ and the above results to get

$$|M_{pp}(\mathbf{q})|^2 \approx \frac{mv_F}{\pi^3}\, \frac{(\Delta\omega_c)^2}{q}\, \cos^2\left(\sqrt{2p}\; ql_c - \frac{\pi}{4}\right) . \tag{3.23}$$

Note that $D''_{11}(\mathbf{q},y)$ consists of two contributions coming from the intra-Landau-level and the inter-Landau-level processes respectively. That is, in the particle-hole bubbles appearing in the $1/N$ expansion (or the RPA approximation) of the gauge field propagator, the particle line and the hole line may carry the same effective-Landau-level index or different indices. For the inter-Landau-level process, there is an excitation gap which is the order of $\Delta\omega_c$. Thus, for $y < \Delta\omega_c$, the intra-Landau-level process is the only contribution to $D''_{11}(\mathbf{q},y)$. As shown before, the intra-Landau-level contribution to a particle-hole bubble gives rise to the $n_F(\xi_l)(1-n_F(\xi_l))$ factor in the gauge field propagator, which becomes exponentially small for $T \ll \Delta\omega_c$. This suggests that $D''_{11}(\mathbf{q},y)$ becomes exponentially small for $y < \Delta\omega_c$ and $T \ll \Delta\omega_c$ so that we can ignore the contribution coming from $y < \Delta\omega_c$ for our purpose. Thus we consider only the contribution coming from the inter-Landau-level process, which appears only above the gap $\Delta\omega_c$. For $\mu/\Delta\omega_c \gg 1$ or the large p approximation, one may argue that the smearing of the discrete spectral function $D''_{11}(\mathbf{q},y)$ of the gauge field propagator, which comes from the Landau-level structure, does not cause any significant change in the global behavior of the response functions. Therefore, we use $D''_{11}(\mathbf{q},y)$ for $\Delta B = 0$ instead of $D''_{11}(\mathbf{q},y)$ for finite ΔB, but a lower cutoff $\Delta\omega_c$ is introduced in the y integral in Eq. (3.20) to mimic the gap in $D''_{11}(\mathbf{q},y)$. Since the precise value of the gap is not known, the numerical coefficient of the final answer to the response function is unreliable, but the functional dependence on $\Delta\omega_c$ is not affected.

The transverse gauge field propagator $D_{11}(\mathbf{q},\omega)$ for $\Delta B = 0$ is given by $1/(-i\gamma\frac{\omega}{q} + \chi q^\eta)$ [6], where $\gamma = \frac{2n_e}{k_F}$, $\chi = \frac{1}{24\pi m} + \frac{V_0}{(2\pi\phi)^2}$ for $\eta = 2$, and $\chi = \frac{V_0}{(2\pi\phi)^2}$ for $\eta \neq 2$. For the large p limit, evaluation of the q integral in Eq. (3.20) gives us

$$\int \frac{d^2q}{(2\pi)^2}\, |M_{pp}(\mathbf{q})|^2\, D''_{11}(\mathbf{q},y) \approx -\frac{mv_F}{8\pi^3}\, \frac{1}{1+\eta}\, \frac{1}{\sin\left(\frac{\pi}{1+\eta}\right)}\, \gamma^{-\frac{\eta-1}{\eta+1}}\, \chi^{-\frac{2}{1+\eta}}\, y^{-\frac{\eta-1}{\eta+1}}\, (\Delta\omega_c)^2 . \tag{3.24}$$

Now we can perform the y integral, yielding

$$\begin{aligned} I &\approx 2\int_{\Delta\omega_c}^{\infty} \frac{dy}{\pi} \sum_{\mathbf{q}} |M_{pp}(\mathbf{q})|^2\, \frac{D''_{11}(\mathbf{q},y)}{y} \\ &= -\frac{mv_F}{4\pi^4}\, \frac{1}{\eta-1}\, \frac{1}{\sin\left(\frac{\pi}{1+\eta}\right)}\, \gamma^{-\frac{\eta-1}{\eta+1}}\, \chi^{-\frac{2}{1+\eta}}\, (\Delta\omega)^{\frac{\eta+3}{\eta+1}} . \end{aligned} \tag{3.25}$$

Therefore, for $\xi_{p+1} = |\xi_p| = \Delta\omega_c/2$, we get

$$\frac{\partial n_1}{\partial \mu} \approx \frac{A(\eta)}{\eta - 1} \frac{m}{\pi} \frac{(\Delta\omega_c)^{\frac{\eta+3}{\eta+1}}}{T^2} , \tag{3.26}$$

where

$$A(\eta) = \frac{v_F}{2\pi^3} \frac{1}{\sin\left(\frac{\pi}{1+\eta}\right)} \gamma^{-\frac{\eta-1}{\eta+1}} \chi^{-\frac{2}{1+\eta}} . \tag{3.27}$$

Combining the result of Eq. (3.26) and that of the free fermions given by Eq. (3.15), we get

$$\frac{\partial n}{\partial \mu} \approx \frac{m(\Delta\omega_c)}{\pi T} e^{-\Delta\omega_c/2T} \left(1 + \frac{A(\eta)}{\eta - 1} \frac{(\Delta\omega_c)^{\frac{2}{1+\eta}}}{T}\right) . \tag{3.28}$$

This is the central result of this chapter.

Note that $A(\eta)$ should be proportional to a small expansion parameter, for example, $1/N$ in a large N generalized model. Thus $1 + \frac{A(\eta)}{\eta-1} \frac{(\Delta\omega_c)^{\frac{2}{1+\eta}}}{T} \approx e^{\frac{A(\eta)}{\eta-1} \frac{(\Delta\omega_c)^{\frac{2}{1+\eta}}}{T}}$ so that the result of Eq. (3.28) is consistent with the renormalized energy gap $\Delta\omega_c^* \approx \Delta\omega_c \left(1 - \frac{2A(\eta)}{\eta-1}(\Delta\omega_c)^{-\frac{\eta-1}{\eta+1}}\right)$ if we write $\partial n/\partial\mu \propto e^{-\Delta\omega_c^*/2T}$. This implies that $m^*/m \approx 1 + \frac{2A(\eta)}{\eta-1}(\Delta\omega_c)^{-\frac{\eta-1}{\eta+1}}$ from $\Delta\omega_c^* = \frac{e\Delta B}{m^* c}$. In particular, for the Coulomb interaction ($\eta = 1$), $\Delta\omega_c^* \approx \Delta\omega_c\,(1 - 2A(\eta = 1)\ \ln\,(\epsilon_F/\Delta\omega_c))$ and $m^*/m \approx 1 + 2A(\eta = 1)\ \ln\,(\epsilon_F/\Delta\omega_c)$. These results were predicted by HLR in terms of a self-consistency argument [6] and are also consistent with the recent work of Stern and Halperin [16].

3.4 Polarization Bubble versus Self-Energy

In the previous sections, we used Eq. (3.11) and the subsequent derivatives of Ω_{toy} to get the correction to the compressibility. There is an alternative way to express Ω_{toy}, which involves the use of the self-energy. That is, Eq. (3.10) can be written as

$$\Omega_{\text{toy}} = -T \sum_{i\omega_n} \sum_{l} \frac{m\Delta\omega_c}{2\pi} \Sigma(\xi_l, i\omega_n)\ G(\xi_l, i\omega_n) , \tag{3.29}$$

where $\omega_n = (2n+1)\pi T$ is the Matsubara frequency and $G(\xi_l, i\omega_n) = \frac{1}{i\omega_n - \xi_l}$. $\Sigma(\xi_l, i\omega_n)$ is the one-loop self-energy correction dressed by the gauge field $\widetilde{D}_{11}$ and is given by the diagrams in Figure 3-5.

We note that Ω_{toy} is finite, whereas Σ is known to be infinite (for $\eta = 2$) at finite temperatures and $\Delta\omega_c = 0$. In this section, we wish to clarify how this apparent difficulty is resolved. Using the spectral representation, we can rewrite Eq. (3.29) as

$$\begin{aligned} \Omega_{\text{toy}} &= \Omega_c + \Omega_d , \\ \Omega_c &= \frac{m\Delta\omega_c}{2\pi} \sum_l \int_{-\infty}^{\infty} \frac{dx}{\pi}\ n_F(x)\ \Sigma''(\xi_l, x)\ G'(\xi_l, x) , \end{aligned}$$

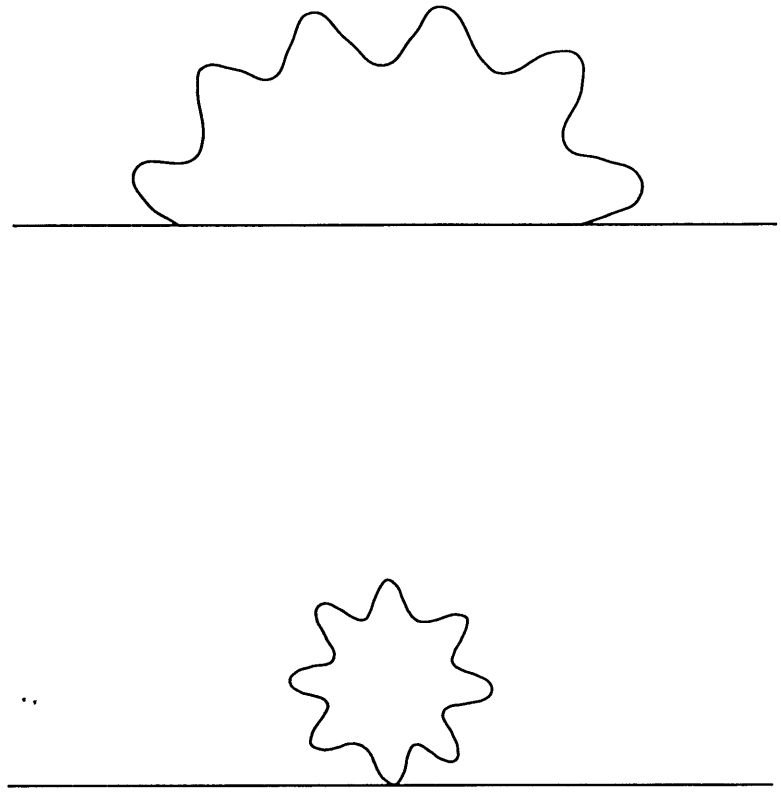


Figure 3-5: The diagrams that represent the lowest order correction to the self-energy of the fermions.

$$\Omega_d = \frac{m\Delta\omega_c}{2\pi} \sum_l \int_{-\infty}^{\infty} \frac{dx}{\pi}\, n_F(x)\, \Sigma^{'}(\xi_l, x)\, G^{''}(\xi_l, x) \ , \tag{3.30}$$

Now we would like to compare two ways of calculating $\Omega_{\rm toy}$. First let us discuss the case of $\Delta\omega_c = 0$. If we use Eq. (3.11), one can show that Ω_a is finite by using $\Pi^{''}_{11}(\mathbf{q}, x) \approx -\gamma x/q$. Suppose that we are going to use only the first diagram of the transeverse part of the polarization bubble in Figure 3-2 (b) to calculate Ω_b. Since the leading contribution of the first diagram to $\Pi^{'}$ is given by n_0/m where n_0 is the density of the free fermions, it can be shown that Ω_b diverges in this case. However, the second diagram also contributes $-n_0/m$ which cancels the constant term of the first diagram. This cancellation is required by the gauge-invariance. As a result, $\Pi^{'} \approx \chi_0 q^2$ with $\chi_0 = \frac{1}{24\pi m}$ so that Ω_b becomes finite. In particular, for the short range interaction ($\eta = 2$), Ω_a and Ω_b give rise to the same contributions with different coefficients.

Next we examine what happens if we use Eq. (3.30) which expresses $\Omega_{\rm toy}$ in terms of the self-energy. For $\Delta\omega_c = 0$, Eq. (3.30) can be rewritten as

$$\begin{aligned}
\Omega_{\rm toy} &= \Omega_c + \Omega_d \ , \\
\Omega_c &= \sum_{\mathbf{k}} \int_{-\infty}^{\infty} \frac{dx}{\pi}\, n_F(x)\, \Sigma^{''}(\xi_{\mathbf{k}}, x)\, G^{'}(\xi_{\mathbf{k}}, x) \ , \\
\Omega_d &= \sum_{\mathbf{k}} \int_{-\infty}^{\infty} \frac{dx}{\pi}\, n_F(x)\, \Sigma^{'}(\xi_{\mathbf{k}}, x)\, G^{''}(\xi_{\mathbf{k}}, x) \ .
\end{aligned} \tag{3.31}$$

As a well known result [11], $\Sigma^{''}(\xi_{\mathbf{k}}, x)$ diverges for $T \neq 0$. Thus we may conclude that Ω_c diverges and this divergence must be cancelled by a similar term in Ω_d. Now one may wonder whether there is any cancellation at the self-energy level especially between the first and the second diagrams in Figure 3-5 as in the case of the polarization bubbles. Since the second diagram generates only the real part, there is no cancellation in $\Sigma^{''}$. For $\Sigma^{'}$, both of the two diagrams contribute. However, one can see that there is no cancellation between the two contributions because of the presence of the additional fermion propagator in the first diagram. We believe that these are the symptoms of the gauge non-invariant nature of the self-energy. In the previous sections, we consider first the polarization bubbles which are gauge-invariant objects. Note that there is an explicit cancellation in this gauge-invariant combination. Therefore, we think that using the polarization bubble makes the gauge-invariance manifest.

Armed with these arguments, we can investigate the $\Delta\omega_c \neq 0$ case. Recalling that the first and the second terms of Eq. (3.7) correspond to the first and the second diagrams of Figure 3-5, we may anticipate a similar cancellation between these two terms as the case of $\Delta\omega_c = 0$. Indeed the f-sum rule, which is given by

$$\sum_{lm} |M_{lm}(\mathbf{q} \to 0)|^2\, \frac{n_F(\xi_l) - n_F(\xi_m)}{\xi_m - \xi_l} = \frac{1}{m}\left(\frac{m\Delta\omega_c}{2\pi} \sum_l n_F(\xi_l)\right) = \frac{n_0}{m} \ , \tag{3.32}$$

allows a cancellation between the first term and the second term in the $q \to 0$, $i\nu_n \to 0$ limit. We use this result in the appendix to estimate various contributions to the compressibility.

3.5 Conclusion

In chapter 2, we showed that the density-density correlation function has a Fermi-liquid form as far as the long wavelength and the low frequency limits are concerned. An important issue is whether this result is compatible with the previous self-consistency treatment based on the one-loop self-energy correction [6] and the present calculation of the energy gap, which are in favor of a divergent effective mass at the half-filling. For a class of Fermi-liquid interaction parameters $f_{\mathbf{pp}'}$, which gives a finite angular average f_{0s}, in chapter 2, we demonstrated that the effective mass is finite if we want to fit the result of the density-density correlation function to the usual Fermi-liquid theory framework [13]. However, it is recently shown that the effect of the divergent effective mass can be cancelled by a contribution from a singular $f_{\mathbf{pp}'}$, in the density-density correlation function for the case of $\eta = 1$[16]. Even though this scenario is quite plausible, it is still not clear whether we are allowed to interpret all physical measurements in terms of the conventional Fermi-liquid theory.

Recently Stern and Halperin [16] calculated the energy gap of the system (for $\eta = 1$) from the one-particle Green's function of the composite fermions in a finite effective magnetic field ΔB. They identified the region where the spectral function vanishes at zero temperature, which is argued to be gauge-invariant, and found an energy gap which is in agreement with the previous self-consistency treatment [6] and the present calculation. The advantage of our calculation is that we directly evaluated the gauge-invariant two particle Green's function, and we could consider the finite temperature situation. We would like to mention that the present perturbative calculation suggests that the perturbation theory for the compressibility breaks down for sufficiently small $\Delta\omega_c$ in the sense that the correction to the energy gap becomes larger than the bare energy gap. In order to understand both of the density-density correlation function and the energy-gap correction, we need a unified framework which can explain both of them in a consisitent way. In the next chapter, we will see that the quantum Boltzmann equation can give us this framework.

Chapter 4

Quantum Boltzmann Equation

4.1 Introduction

As mentioned in chapter 3, it is necessary to reconcile the result of the density-density correlation function at $\nu = 1/2$ and the correction to the energy gap near $\nu = 1/2$ calculated from the compressibility. One resolution of the problem is suggested by Stern and Halperin [16] within the usual Landau-Fermi-liquid theory framework. The idea is that both of the effective mass and the Landau-interaction-function are singular in such a way that they cancel each other in the density-density correlation function. Recently, Stern and Halperin [16] put forward this idea and construct a Fermi-liquid-theory of the fermion-gauge system in the case of Coulomb interaction. Even though the use of the Landau-Fermi-liquid theory or equivalently the existence of the well defined quasi-particles can be *marginally* justified in the case of the Coulomb interacion, we feel that it is necessary to construct a more general framework which applies to the arbitrary two-particle interaction ($1 < \eta \leq 2$ as well as $\eta = 1$) and allows us check the validity of the Fermi liquid theory and to judge when the divergent mass shows up. In particular, it is worthwhile to provide a unified picture for understanding the previous theoretical studies.

In the usual Fermi-liquid theory, the QBE of the quasi-particles provides the useful informations about the low lying excitations of the system. Our objective is to construct a similar QBE which describes all the low energy physics of the composite fermion system. One important difficulty we are facing here is that we cannot assume the existence of the quiasi-particles *a priori* in the derivation of the QBE even though the conventional derivation of the QBE of the Fermi-liquid theory relies on the existence of these quasi-particles. Following closely the work of Prange and Kadanoff [32] about the electron-phonon system, where there is also no well defined quasi-particle at temperatures high compared with the Debye temperature, we concentrate on a generalized Fermi surface displacement which, in our case, corresponds to the local variation of the chemical potential in momentum space. Due to the non-existence of a well defined quasiparticle, the usual distribution function n_k in the momentum space cannot be described by a closed equation of motion. However we will see later that the generalized Fermi surface displacement does satisfy a closed equation of motion.

This equation of motion will be also called as QBE.

We use the non-equilibrium Green's function technique [32, 33, 34, 35] to derive the new QBE and calculate the generalized Landau-interaction-function which has the frequency dependence as well as the usual angular dependence due to the retarded nature of the gauge interaction. The QBE at $\nu = 1/2$ consists of three parts. One is the contribution from the self-energy correction which gives the singular mass correction, the other one comes from the generalized Landau-interaction-function, and finally it contains the collision integral. These quantities are calculated to the lowest order in the coupling to the gauge field.

By studing the dynamical properties of the collective modes using the QBE, we find that the smooth fluctuations of the Fermi surface (or the small angular momentum modes) show the usual Fermi-liquid behavior, while the rough fluctuations (or the large angular momentum modes) show the singular behavior determined by the singular self-energy correction. Here the angular momentum is the conjugate variable of the angle measured from a given direction in momentum space. There is a forward scattering cancellation between the singular self-energy correction and the singular (generalized) Landau-interaction-function and a similar cancellation exists in the collision integral as far as the small angular momentum modes $l < l_c$ ($l_c \propto \Omega^{-\frac{1}{1+\eta}}$, where Ω is the small external frequency) are concerned. However, in the case of the large angular momentum modes $l > l_c$, the contribution from the Landau-interaction-function becomes very small so that the self-energy correction dominates and the collision integral also cannot be ignored in general. In this case the behaviors of the low lying modes are very different from those in the Fermi liquids.

If we ignore the collision integral, it can be shown that the system has a lot of collective modes between $\Omega \propto q^{\frac{1+\eta}{2}}$ $(1 < \eta \leq 2)$, $\Omega \propto q/|\ln q|$ $(\eta = 1)$ and $\Omega = v_F q$ while there is the particle-hole continuum below $\Omega \propto q^{\frac{1+\eta}{2}}$ $(1 < \eta \leq 2)$, $\Omega \propto q/|\ln q|$ $(\eta = 1)$. The distinction between these two types of low lying excitations are obscured by the existence of the collision integral.

From the above results, we see that the density-density and the current-current correlation functions, being dominated by the small angular momentum modes $l < l_c$, show the usual Fermi-liquid behavior. On the other hand, the energy gap away from $\nu = 1/2$ is determined by the behaviors of the large angular momentum modes $l > l_c$ so that the singular mass correction shows up in the energy gap of the system.

The outline of this chapter is as follows. In section 4.2, we explain the way we contruct the QBE without assuming the existence of the quasi-particles. In section 4.3, the QBE for the generalized distribution function is derived for $\Delta B = 0$. In section 4.4, we construct the QBE for the generalized Fermi surface displacement for $\Delta B = 0$. We also determine the generalized Landau-interaction-function and discuss its consequences. In section 4.5, The QBE in the presence of a small ΔB is constructed and the energy gap of the system is determined. In section 4.6, We discuss the collective excitations of the system for the cases of $\Delta B = 0$ and $\Delta B \neq 0$. We conclude the chapter and discuss the implications of our results to experiments in section 4.7. We concentrate on the zero temperature case in the main text and provide the derivation of the QBE at finite temperatures in the appendix B, which requires some special treatments compared to the zero temperature counterpart.

4.2 The Quantum Boltzmann Equation in the Absence of the Quasi-Particles

Before explaining the way we construct the QBE for the fermion-gauge-field system in which there is no well defined Landau-quasi-particle in general, we review the usual derivation of the QBE for the Fermi-liquid with well defined quasi-particles [33, 34, 35]. The QBE is nothing but the equation of motion of the fermion distribution function. Therefore, it can be derived from the equation of motion of the non-equilibrium one-particle Green's function. Following Kadanoff and Baym [33], let us consider the following one-particle Green's function.

$$G^{<}(x_1, x_2) = i\langle\psi^\dagger(x_2)\psi(x_1)\rangle \ , \tag{4.1}$$

where $x_1 = (\mathbf{r}_1, t_1)$ and $x_2 = (\mathbf{r}_2, t_2)$. At non-equilibrium, $G^{<}(x_1, x_2)$ does not satisfy the translational invariance in space-time so that it cannot be written as $G^{<}(x_1 - x_2)$. By the following change of variables

$$(\mathbf{r}_{\rm rel}, t_{\rm rel}) = x_1 - x_2 \ \text{ and } \ (\mathbf{r}, t) = (x_1 + x_2)/2 \ , \tag{4.2}$$

$G^{<}(x_1, x_2)$ can be written as

$$G^{<}(\mathbf{r}_{\rm rel}, t_{\rm rel}; \mathbf{r}, t) = i\langle\psi^\dagger(\mathbf{r} - \frac{\mathbf{r}_{\rm rel}}{2}, t - \frac{t_{\rm rel}}{2}) \ \psi(\mathbf{r} + \frac{\mathbf{r}_{\rm rel}}{2}, t + \frac{t_{\rm rel}}{2})\rangle \ . \tag{4.3}$$

Using the Fourier transformation for the relative coordinates $t_{\rm rel}$ and $\mathbf{r}_{\rm rel}$, we get $G^{<}(\mathbf{p}, \omega; \mathbf{r}, t)$. At equilibrium, $G^{<}$ can be written as [33, 34, 35]

$$G_0^{<}(\mathbf{p}, \omega) = i f_0(\omega) A(\mathbf{p}, \omega) \ , \tag{4.4}$$

where $f_0(\omega) = 1/(e^{\omega/T} + 1)$ is the equilibrium Fermi distribution function and (Σ^R is the retarded self-energy)

$$A(\mathbf{p}, \omega) = \frac{-2 \ {\rm Im} \ \Sigma^R \ (\mathbf{p}, \omega)}{(\omega - \xi_p - {\rm Re} \ \Sigma^R(\mathbf{p}, \omega))^2 + ({\rm Im} \ \Sigma^R(\mathbf{p}, \omega))^2} \ . \tag{4.5}$$

In the usual Fermi-liquid theory, Im $\Sigma^R \ll \omega$ so that $A(\mathbf{p}, \omega)$ is a peaked function of ω around $\omega = \xi_p + {\rm Re} \ \Sigma^R$. In this case, the equilibrium spectral function can be taken as [33, 34, 35]

$$A(\mathbf{p}, \omega) = 2\pi\delta(\omega - \xi_p - {\rm Re} \ \Sigma^R(\mathbf{p}, \omega)) \ . \tag{4.6}$$

Using this property, if the system is not far away from the equilibrium, one can construct a closed equation for the fermion distribution function $f(\mathbf{p}, \mathbf{r}, t)$ [33, 34, 35], which is the QBE. The linearized QBE of $\delta f(\mathbf{p}, \mathbf{r}, t) = f(\mathbf{p}, \mathbf{r}, t) - f_0(\mathbf{p})$, where $f_0(\mathbf{p})$

is the equilibrium distribution function, is the QBE of the quasi-particles in the Fermi-liquid theory. From this QBE, the equation of motion for the Fermi surface deformation, which is defined as [33, 34, 35]

$$\nu(\theta, \mathbf{r}, t) = \int d|\mathbf{p}| \; \delta f(\mathbf{p}, \mathrm{r}, t) \;, \tag{4.7}$$

can be also constructed.

In the case of the fermion-gauge-field system, as mentioned in the introduction, Im $\Sigma^R(\omega)$ is larger than ω ($1 < \eta \leq 2$) or comparable to ω ($\eta = 1$), *i.e.*, strictly speaking, there is no well defined Landau-quasi-particle from the viewpoint of perturbation theory. However, Stern and Halperin [16] showed that, within a self-consistent treatment, the Fermi-liquid theory can be barely applied to the case of Coulomb interaction in the sense that Re Σ^R is logarithmically larger than Im Σ^R. Note that, in general, $A(\mathbf{p}, \omega)$ at equilibrium is not a peaked function of ω anymore in the fermion-gauge-field system. Because of this, $f(\mathbf{p}, \mathbf{r}, t)$ does not satisfy a closed equation of motion even near the equilibrium. However, if Σ^R is only a function of ω, $A(\mathbf{p}, \omega)$ is still a well peaked function of ξ_p around $\xi_p = 0$ for sufficiently small ω [32]. This observation leads us to define the following generalized distribution function [32]

$$f(\theta, \omega; \mathbf{r}, t) = -i \int \frac{d\xi_p}{2\pi} \; G^<(\mathbf{p}, \omega; \mathbf{r}, t) \;, \tag{4.8}$$

where θ is the angle between $\mathbf{p}$ and a given direction. The linearized quantum Boltzmann equation for $\delta f(\theta, \omega; \mathbf{r}, t) = f(\theta, \omega; \mathbf{r}, t) - f_0(\omega)$ can be derived, which is analogous to the QBE of the quasi-particles in the usual Fermi-liquid theory. From this QBE, one can also construct the equation of motion for the generalized Fermi surface displacement [32]

$$u(\theta, \mathbf{r}, t) = \int \frac{d\omega}{2\pi} \; \delta f(\theta, \omega; \mathbf{r}, t) \tag{4.9}$$

which corresponds to the variation of the local chemical potential in the momentum space. This object can be still well defined even in the absence of the sharp Fermi surface. This is because one can always define a chemical potential in each angle θ, which is the energy required to put an additional fermion in the direction labeled by θ in the momentum space. In the next section, we derive the linearized QBE for the generalized distribution function $\delta f(\theta, \omega; \mathbf{r}, t)$.

4.3 Quantum Boltzmann Equation for the Generalized Distribution Function

In the non-equilibrium Green's function formulation, the following matrices of the Green's function and the self-energy satisfy the Dyson's equation [35]

$$\tilde{G} = \begin{pmatrix} G_t & -G^< \\ G^> & -G_{\bar{t}} \end{pmatrix} \quad \text{and} \quad \tilde{\Sigma} = \begin{pmatrix} \Sigma_t & -\Sigma^< \\ \Sigma^> & -\Sigma_{\bar{t}} \end{pmatrix}, \tag{4.10}$$

where

$$
\begin{aligned}
G^{>}(x_1,x_2) &= -i\langle\psi(x_1)\psi^{\dagger}(x_2)\rangle, \\
G^{<}(x_1,x_2) &= i\langle\psi^{\dagger}(x_1)\psi(x_2)\rangle, \\
G_t(x_1,x_2) &= \Theta(t_1-t_2)G^{>}(x_1,x_2)+\Theta(t_2-t_1)G^{<}(x_1,x_2), \\
G_{\bar{t}}(x_1,x_2) &= \Theta(t_2-t_1)G^{>}(x_1,x_2)+\Theta(t_1-t_2)G^{<}(x_1,x_2),
\end{aligned}
\tag{4.11}
$$

and $\Sigma^{>}, \Sigma^{<}, \Sigma_t, \Sigma_{\bar{t}}$ are the associated self-energies. $\Theta(t)=1$ for $t>0$ and zero for $t<0$. G^R (retarded) and G^A (advanced) Green's functions can be expressed in terms of G_t (time-ordered), $G_{\bar{t}}$ (antitime-ordered), $G^{<}$, $G^{>}$ as follows.

$$
\begin{aligned}
G^R &= G_t - G^{<} = G^{>} - G_{\bar{t}}\ , \\
G^A &= G_t - G^{>} = G^{<} - G_{\bar{t}}\ .
\end{aligned}
\tag{4.12}
$$

Similarly, Σ^R and Σ^A are given by

$$
\begin{aligned}
\Sigma^R &= \Sigma_t - \Sigma^{<} = \Sigma^{>} - \Sigma_{\bar{t}}\ , \\
\Sigma^A &= \Sigma_t - \Sigma^{>} = \Sigma^{<} - \Sigma_{\bar{t}}\ .
\end{aligned}
\tag{4.13}
$$

The matrix Green's function satisfies the following equations of motion

$$
\begin{aligned}
\left[i\frac{\partial}{\partial t_1} - H_0(\mathbf{r}_1)\right]\tilde{G}(x_1,x_2) &= \delta(x_1-x_2)\tilde{I} + \int dx_3\ \tilde{\Sigma}(x_1,x_3)\tilde{G}(x_3,x_2), \\
\left[-i\frac{\partial}{\partial t_2} - H_0(\mathbf{r}_2)\right]\tilde{G}(x_1,x_2) &= \delta(x_1-x_2)\tilde{I} + \int dx_3\ \tilde{G}(x_1,x_3)\tilde{\Sigma}(x_3,x_2),
\end{aligned}
\tag{4.14}
$$

where

$$
H_0(\mathbf{r}_1) = -\frac{1}{2m}\left(\frac{\partial}{\partial \mathbf{r}_1}\right)^2 - \mu \ \text{ and }\ H_0(\mathbf{r}_2) = -\frac{1}{2m}\left(\frac{\partial}{\partial \mathbf{r}_2}\right)^2 - \mu\ . \tag{4.15}
$$

For our purpose, we need only the equation of motion for $G^{<}$

$$
\begin{aligned}
\left[i\frac{\partial}{\partial t_1} - H_0(\mathbf{r}_1)\right]G^{<}(x_1,x_2) &= \int dx_3\,[\Sigma_t(x_1,x_3)G^{<}(x_3,x_2) \\
&\quad - \Sigma^{<}(x_1,x_3)G_{\bar{t}}(x_3,x_2)]\ , \\
\left[-i\frac{\partial}{\partial t_2} - H_0(\mathbf{r}_2)\right]G^{<}(x_1,x_2) &= \int dx_3\,[G_t(x_1,x_3)\Sigma^{<}(x_3,x_2) \\
&\quad - G^{<}(x_1,x_3)\Sigma_{\bar{t}}(x_3,x_2)]\ .
\end{aligned}
\tag{4.16}
$$

Taking the difference of the two equations of Eq. (4.16), and using the following relations

$$
\begin{aligned}
G_t &= \mathrm{Re}\ G^R + \frac{1}{2}(G^{<}+G^{>})\ , \\
G_{\bar{t}} &= \frac{1}{2}(G^{<}+G^{>}) - \mathrm{Re}\ G^R\ ,
\end{aligned}
\tag{4.17}
$$

we get

$$
\left[i\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} + \frac{1}{2m}\left(\frac{\partial}{\partial \mathbf{r}_1}\right)^2 - \frac{1}{2m}\left(\frac{\partial}{\partial \mathbf{r}_2}\right)^2 \right] G^<(x_1,x_2)
$$
$$
\begin{aligned}
= \int dx_3\ [\ & \mathrm{Re}\ \Sigma^R(x_1,x_3)G^<(x_3,x_2) + \Sigma^<(x_1,x_3)\mathrm{Re}\ G^R(x_3,x_2) \\
& -\mathrm{Re}\ G^R(x_1,x_3)\Sigma^<(x_3,x_2) - G^<(x_1,x_3)\mathrm{Re}\ \Sigma^R(x_3,x_2) \\
& +\frac{1}{2}\Sigma^>(x_1,x_3)G^<(x_3,x_2) - \frac{1}{2}\Sigma^<(x_1,x_3)G^>(x_3,x_2) \\
& -\frac{1}{2}G^>(x_1,x_3)\Sigma^<(x_3,x_2) + \frac{1}{2}G^<(x_1,x_3)\Sigma^>(x_3,x_2)\ ]\ .
\end{aligned} \tag{4.18}
$$

Near equilibrium, one can linearize this equation assuming that $\delta\tilde{G} = \tilde{G} - \tilde{G}_0$ and $\delta\tilde{\Sigma} = \tilde{\Sigma} - \tilde{\Sigma}_0$ are small, where $\tilde{G}_0$ and $\tilde{\Sigma}_0$ are matrices of the equilibrium Green's function and the self-energy. The Fourier transform $\tilde{G}(p_1,p_2)$ ($p_1 = (\mathbf{p}_1,\omega_1)$, $p_2 = (\mathbf{p}_2,\omega_2)$) of $\tilde{G}(x_1,x_2)$ can be written in terms of the new variables defined by

$$
p = (\mathbf{p},\omega) = (p_1 - p_2)/2 \quad \text{and} \quad q = (\mathbf{q},\Omega) = p_1 + p_2\ . \tag{4.19}
$$

Using these variables, the Fourier transformed linearized equation of $\delta G^<(p,q)$ can be written as

$$
\begin{aligned}
& [\ \Omega - v_F|\mathbf{q}|\ \cos\theta_{\mathbf{pq}}\ ]\ \delta G^<(p,q) \\
& -[\ \mathrm{Re}\ \Sigma_0^R(p+q/2) - \mathrm{Re}\ \Sigma_0^R(p-q/2)\ ]\ \delta G^<(p,q) \\
& +[\ G_0^<(p+q/2) - G_0^<(p-q/2)\ ]\ \delta(\mathrm{Re}\ \Sigma^R(p,q)) \\
& -[\ \Sigma_0^<(p+q/2) - \Sigma_0^<(p-q/2)\ ]\ \delta(\mathrm{Re}\ G^R(p,q)) \\
& +[\ \mathrm{Re}\ G_0^R(p+q/2) - \mathrm{Re}\ G_0^R(p-q/2)\ ]\ \delta\Sigma^<(p,q) \\
& = G_0^<(p)\ \delta\Sigma^>(p,q) + \Sigma_0^>(p)\ \delta G^<(p,q) \\
& \quad -G_0^>(p)\ \delta\Sigma^<(p,q) - \Sigma_0^<(p)\ \delta G^>(p,q)\ ,
\end{aligned} \tag{4.20}
$$

where $\theta_{\mathbf{pq}}$ is the angle between $\mathbf{p}$ and $\mathbf{q}$. In the presence of an external potential $U(q)$, one should add a term $U(q)\ [\ G_0^<(p+q/2) - G_0^<(p-q/2)\ ]$ in the left hand side of Eq. (4.20).

We next check that this expression is equivalent to the usual QBE for $\delta G^<(\mathbf{p},\omega;\mathbf{r},t)$, where $\mathbf{r}$ and t are conjugate variables of $\mathbf{q}$ and Ω. Note that

$$
F(p+q/2) - F(p-q/2) \approx \mathbf{q}\cdot\frac{\partial F}{\partial \mathbf{p}} + \Omega\frac{\partial F}{\partial \omega}\ , \tag{4.21}
$$

for small $|\mathbf{q}|$ and Ω. From Eq. (linear) and Eq. (4.21), one can check that $\delta G^<(\mathbf{p},\omega;\mathbf{r},t)$, which is the Fourier transform of $\delta G^<(p,q)$, satisfies the following equation.

$$
\begin{aligned}
& [\ \omega - p^2/2m, \delta G^<(\mathbf{p},\omega;\mathbf{r},t)\ ] \\
& -[\ \mathrm{Re}\ \Sigma_0^R(\mathbf{p},\omega), G^<(\mathbf{p},\omega;\mathbf{r},t)\ ] - [\ \delta(\mathrm{Re}\ \Sigma^R(\mathbf{p},\omega)), G_0^<(\mathbf{p},\omega)\ ] \\
& -[\ \Sigma_0^<(\mathbf{p},\omega), \delta(\mathrm{Re}\ G^R(\mathbf{p},\omega;\mathbf{r},t)\ ] - [\ \delta\Sigma^<(\mathbf{p},\omega;\mathbf{r},t), \mathrm{Re}\ G_0^R(\mathbf{p},\omega)\ ] \\
& = G_0^<(\mathbf{p},\omega)\ \delta\Sigma^>(\mathbf{p},\omega;\mathbf{r},t) + \Sigma_0^>(\mathbf{p},\omega)\ \delta G^<(\mathbf{p},\omega;\mathbf{r},t) \\
& \quad -G_0^>(\mathbf{p},\omega)\ \delta\Sigma^<(\mathbf{p},\omega;\mathbf{r},t) - \Sigma_0^<(\mathbf{p},\omega)\ \delta G^>(\mathbf{p},\omega;\mathbf{r},t)\ ,
\end{aligned} \tag{4.22}
$$

where $[X, Y]$ is the Poisson braket

$$[X, Y] = \frac{\partial X}{\partial \omega}\frac{\partial Y}{\partial t} - \frac{\partial X}{\partial t}\frac{\partial Y}{\partial \omega} - \frac{\partial X}{\partial \mathbf{p}} \cdot \frac{\partial Y}{\partial \mathbf{r}} + \frac{\partial X}{\partial \mathbf{r}} \cdot \frac{\partial Y}{\partial \mathbf{p}} . \tag{4.23}$$

Note that this equation is just the linearized version of the usual QBE for $G^<(\mathbf{p}, \omega; \mathbf{r}, t)$ given by [33, 34, 35]

$$\begin{aligned}
&[\ \omega - p^2/2m - \text{Re}\ \Sigma^R(\mathbf{p}, \omega; \mathbf{r}, t), G^<(\mathbf{p}, \omega; \mathbf{r}, t)\ ] \\
&-[\ \Sigma^<(\mathbf{p}, \omega; \mathbf{r}, t), \text{Re}\ G^R(\mathbf{p}, \omega; \mathbf{r}, t)\ ] \\
&= \Sigma^>(\mathbf{p}, \omega; \mathbf{r}, t) G^<(\mathbf{p}, \omega; \mathbf{r}, t) - G^>(\mathbf{p}, \omega; \mathbf{r}, t) \Sigma^<(\mathbf{p}, \omega; \mathbf{r}, t) .
\end{aligned} \tag{4.24}$$

We directly deal with Eq. (4.20) in momentum space $(\mathbf{q}, \Omega)$ rather than the long time, long wave length expansion in real space $(\mathbf{r}, t)$ given by Eq. (4.22). For simplicity, we assume that the gauge field is in equilibrium. The non-equilibrium one-loop self-energy correction, which is given by the diagram in Figure 1-2, can be written as [32, 34, 35]

$$\begin{aligned}
\Sigma^<(\mathbf{p}, \omega) &= \sum_{\mathbf{q}} \int_0^\infty \frac{d\nu}{\pi} \left| \frac{\mathbf{p} \times \hat{\mathbf{q}}}{m} \right|^2 \text{Im}\ D_{11}(\mathbf{q}, \nu) \\
&\quad \times [\ (n_0(\nu) + 1) G^<(\mathbf{p} + \mathbf{q}, \omega + \nu) + n_0(\nu) G^<(\mathbf{p} + \mathbf{q}, \omega - \nu)\ ] , \\
\Sigma^>(\mathbf{p}, \omega) &= \sum_{\mathbf{q}} \int_0^\infty \frac{d\nu}{\pi} \left| \frac{\mathbf{p} \times \hat{\mathbf{q}}}{m} \right|^2 \text{Im}\ D_{11}(\mathbf{q}, \nu) \\
&\quad \times [\ n_0(\nu) G^>(\mathbf{p} + \mathbf{q}, \omega + \nu) + (n_0(\nu) + 1) G^>(\mathbf{p} + \mathbf{q}, \omega - \nu)\ ]
\end{aligned} \tag{4.25}$$

where $n_0(\nu) = 1/(e^{\nu/T} - 1)$ is the equilibrium boson distribution function. The real part of the retarded self-energy is given by

$$\begin{aligned}
\text{Re}\ \Sigma^R(\mathbf{p}, \omega; \mathbf{q}, \Omega) &= -\int \frac{d\omega'}{\pi}\ \mathcal{P} \frac{\text{Im}\ \Sigma^R(\mathbf{p}, \omega'; \mathbf{q}, \Omega)}{\omega - \omega'} \\
&= -\int \frac{d\omega'}{2\pi i}\ \mathcal{P} \frac{\Sigma^>(\mathbf{p}, \omega'; \mathbf{q}, \Omega) - \Sigma^<(\mathbf{p}, \omega'; \mathbf{q}, \Omega)}{\omega - \omega'} ,
\end{aligned} \tag{4.26}$$

where $\mathcal{P}$ represents the principal value and $\text{Im}\ \Sigma^R = \frac{1}{2i}(\Sigma^> - \Sigma^<)$ is used. The same relations hold for the Green's function G^R,

$$\text{Re}\ G^R(\mathbf{p}, \omega; \mathbf{q}, \Omega) = -\int \frac{d\omega'}{2\pi i}\ \mathcal{P} \frac{G^>(\mathbf{p}, \omega'; \mathbf{q}, \Omega) - G^<(\mathbf{p}, \omega'; \mathbf{q}, \Omega)}{\omega - \omega'} \tag{4.27}$$

and $\text{Im}\ G^R = \frac{1}{2i}(G^> - G^<)$.

At equilibrium, the Green's functions $G^<$, $G^>$ can be written as [33, 34, 35]

$$\begin{aligned}
G^<(\mathbf{p}, \omega) &= i f_0(\omega) A(\mathbf{p}, \omega) , \\
G^>(\mathbf{p}, \omega) &= -i(1 - f_0(\omega)) A(\mathbf{p}, \omega) ,
\end{aligned} \tag{4.28}$$

where $A(\mathbf{p}, \omega)$ is given by Eq. (4.5). From these relations, the one-loop self-energy

Σ_0^R at equilibrium can be written as

$$\Sigma_0^R(\mathbf{p},\omega) = \sum_{\mathbf{q}} \int_0^\infty \frac{d\nu}{\pi} \left| \frac{\mathbf{p} \times \hat{\mathbf{q}}}{m} \right|^2 \left[\frac{1 + n_0(\nu) - f_0(\xi_{\mathbf{p}+\mathbf{q}})}{\omega + i\delta - \xi_{\mathbf{p}+\mathbf{q}} - \nu} + \frac{n_0(\nu) + f_0(\xi_{\mathbf{p}+\mathbf{q}})}{\omega + i\delta - \xi_{\mathbf{p}+\mathbf{q}} + \nu} \right] \tag{4.29}$$

As emphasized in the previous section, if the self-energy depends only on the frequency ω, $A(\mathbf{p},\omega)$ at equilibrium is a peaked function of ξ_p. Therefore, as far as the system is not far away from the equilibrium, the generalized distribution function $f(\theta_{\mathbf{pq}}, \omega; \mathbf{q}, \Omega)$, which is given by the following relations, can be well defined at zero temperature [32]:

$$\begin{aligned} \int \frac{d\xi_p}{2\pi} \left[-iG^<(\mathbf{p},\omega;\mathbf{q},\Omega)\right] &\equiv f(\theta_{\mathbf{pq}},\omega;\mathbf{q},\Omega) \ , \\ \int \frac{d\xi_p}{2\pi} \left[iG^>(\mathbf{p},\omega;\mathbf{q},\Omega)\right] &\equiv 1 - f(\theta_{\mathbf{pq}},\omega;\mathbf{q},\Omega) \ , \end{aligned} \tag{4.30}$$

where $\theta_{\mathbf{pq}}$ is the angle between $\mathbf{p}$ and $\mathbf{q}$.

The extension to the case of finite temperatures requires special care because, even at equilibrium, Im $\Sigma_0^R(\mathbf{p},\omega)$ is known to be divergent [11] so that $A(\mathbf{p},\omega)$, $G_0^<$, and $G_0^>$ at equilibrium are not well defined. Therefore, the non-equilibrium $G^<$ and $G^>$ are also not well defined near equilibrium. In order to resolve this problem, let us first separate the gauge field fluctuations into two parts, *i.e.*, $\mathbf{a}(\mathbf{q},\nu) \equiv \mathbf{a}_-(\mathbf{q},\nu)$ for $\nu < T$ and $\mathbf{a}(\mathbf{q},\nu) \equiv \mathbf{a}_+(\mathbf{q},\nu)$ for $\nu > T$, then examine the effects of $\mathbf{a}_+$, $\mathbf{a}_-$ separately. The classical fluctuation $\mathbf{a}_-$ of the gauge field can be regarded as a vector potential which corresponds to a static but spatially varying magnetic field $\mathbf{b}_- = \nabla \times \mathbf{a}_-$. In order to remove the divergence in the self-energy, one can consider the one-particle Green's function $\tilde{G}_- \equiv \tilde{G}(\mathbf{P}_-,\omega;\mathbf{r},t)$ as a function of a new variable $\mathbf{P}_- = \mathbf{p} - \mathbf{a}_-$. Since we effectively separate out $\mathbf{a}_-$ fluctuations, the self-energy, which appears in the equation of motion given by Eq. (4.14), should contain only $\mathbf{a}_+$ fluctuations and is free of divergences. Therefore, $\delta G_-^< \equiv \delta G^<(\mathbf{P}_-,\omega;\mathbf{r},t)$ is well defined and its equation of motion is given by the Fourier transform of Eq. (4.20) with the following replacement. In the first place, the variable $\mathbf{p}$ should be changed to a new variable $\mathbf{P}_- = \mathbf{p} - \mathbf{a}_-$. Secondly, the self-energy $\tilde{\Sigma}$ should be changed to $\tilde{\Sigma}_+$ which contains now only $\mathbf{a}_+$ fluctuations. Finally, the equation of motion contains a term which depends on $\mathbf{b}_-$. We argued in the appendix B that ignoring this term does not affect the physical interpretations of the QBE, which will appear in sections 4.4, 4.5, and 4.6. We provide the details of the anaysis for the finite temperature case in the appendix. From now on, we will adopt the notation that $G^<$ should be understood as $G_-^<$ for finite temperatures. For example, the generalized distribution function at finite temperatures is given by Eq. (4.30) with the replacement that $G^<, G^> \to G_-^<, G_-^>$. The same type of abuse of notation applies to the self-energy, where only $\mathbf{a}_+$ fluctuations should be included, *i.e.* the QBE is valid at finite T, provided that the lower cutoff T is introduced for the frequency integrals.

In Eq. (4.25), one can change the variables such that $\mathbf{p}' = \mathbf{p} + \mathbf{q}$ and $\omega' = \omega + \nu$. The gauge field propagator can be written in terms of the new variables as $D_{11}(\mathbf{q},\nu) =$

$D_{11}(\mathbf{p}'-\mathbf{p},\omega'-\omega)$, where $(\mathbf{p},\omega)$ and $(\mathbf{p}',\omega')$ represent the incoming and outgoing fermions. Assuming that $|\mathbf{p}| \approx |\mathbf{p}'| \approx k_F$ and using $|\mathbf{p}'-\mathbf{p}| \approx k_F|\theta_{\mathbf{p'q}}-\theta_{\mathbf{pq}}|$, we get $D_{11}(\mathbf{q},\nu) \approx D_{11}(k_F|\theta_{\mathbf{p'q}}-\theta_{\mathbf{pq}}|,\omega'-\omega)$. Using the above results and the fact that $G^<$ and $G^>$ are well peaked functions of ξ_p near the equilibrium, Re Σ^R can be written as

$$\mathrm{Re}\ \Sigma^R = N(0)\int\frac{d\theta_{\mathbf{p'q}}}{2\pi}\int d\omega'\ v_F^2\ \mathrm{Re}\ D_{11}(k_F|\theta_{\mathbf{p'q}}-\theta_{\mathbf{pq}}|,\omega'-\omega)\ f(\theta_{\mathbf{p'q}},\omega';\mathbf{q},\Omega)\ , \tag{4.31}$$

where $N(0)=\frac{m}{2\pi}$ is the density of state. Since we assume that the gauge field is at equilibrium, $\delta(\mathrm{Re}\ \Sigma^R)$, which is the deviation from the equilibrium, can be written as

$$\delta(\mathrm{Re}\ \Sigma^R) = N(0)\int\frac{d\theta_{\mathbf{p'q}}}{2\pi}\int d\omega'\ v_F^2\ \mathrm{Re}\ D_{11}(k_F|\theta_{\mathbf{p'q}}-\theta_{\mathbf{pq}}|,\omega'-\omega)\ \delta f(\theta_{\mathbf{p'q}},\omega';\mathbf{q},\Omega)\ . \tag{4.32}$$

We also assume that the non-equilibrium self-energy depends only on ω as that of the equilibrium case, which is plausible as far as the system is not far away from the equilibrium. In order to get the equation for $f(\theta_{\mathbf{pq}},\omega;\mathbf{q},\Omega)$, we perform $\int d\xi_p/2\pi$ integral on both sides of the Eq. (4.20). Note that

$$\begin{aligned}
&\int\frac{d\xi_p}{2\pi}\ \mathrm{Re}\ G^R(\mathbf{p},\omega';\mathbf{q},\Omega)\\
&=\int\frac{d\omega'}{2\pi}\ \mathcal{P}\frac{(1-f(\theta_{\mathbf{pq}},\omega';\mathbf{q},\Omega))+f(\theta_{\mathbf{pq}},\omega';\mathbf{q},\Omega)}{\omega-\omega'}\\
&=\int\frac{d\omega'}{2\pi}\ \mathcal{P}\frac{1}{\omega-\omega'}=0\ .
\end{aligned} \tag{4.33}$$

Thus the fourth and the fifth terms in the left hand side of the QBE (given by Eq. (4.20)) vanish after $\int d\xi_p/2\pi$ integration. After this integral, using Eqs. (4.26), (4.30) and (4.32), the remaining parts of the Eq. (4.20) can be written as ($\delta f(\theta_{\mathbf{pq}},\omega)\equiv \delta f(\theta_{\mathbf{pq}},\omega;\mathbf{q},\Omega)$)

$$\begin{aligned}
&[\ \Omega - v_F q\cos\theta_{\mathbf{pq}}\ ]\ \delta f(\theta_{\mathbf{pq}},\omega)\\
&-N(0)\int\frac{d\theta_{\mathbf{p'q}}}{2\pi}\int d\omega'\ v_F^2\ \mathrm{Re}\ D_{11}(k_F|\theta_{\mathbf{p'q}}-\theta_{\mathbf{pq}}|,\omega'-\omega)\\
&\quad\times[\ f_0(\omega'+\Omega/2)-f_0(\omega'-\Omega/2)\ ]\ \delta f(\theta_{\mathbf{pq}},\omega)\\
&+N(0)\int\frac{d\theta_{\mathbf{p'q}}}{2\pi}\int d\omega'\ v_F^2\ \mathrm{Re}\ D_{11}(k_F|\theta_{\mathbf{p'q}}-\theta_{\mathbf{pq}}|,\omega'-\omega)\\
&\quad\times[\ f_0(\omega+\Omega/2)-f_0(\omega-\Omega/2)\ ]\ \delta f(\theta_{\mathbf{p'q}},\omega')\\
&=N(0)\int d\theta_{\mathbf{p'q}}\int_0^\infty\frac{d\nu}{\pi}\int d\omega'\ v_F^2\ \mathrm{Im}\ D_{11}(k_F|\theta_{\mathbf{p'q}}-\theta_{\mathbf{pq}}|,\nu)\\
&\quad\times[\ \delta(\omega'-\omega+\nu)\ [\ \delta f(\theta_{\mathbf{pq}},\omega)\ (1-f_0(\omega')+n_0(\nu))\\
&\quad-\delta f(\theta_{\mathbf{p'q}},\omega')\ (f_0(\omega)+n_0(\nu))\ ]\\
&\quad-\delta(\omega'-\omega-\nu)\ [\ \delta f(\theta_{\mathbf{p'q}},\omega')\ (1-f_0(\omega)+n_0(\nu))\\
&\quad-\delta f(\theta_{\mathbf{pq}},\omega)(f_0(\omega')+n_0(\nu))\ ]\ ]\ .
\end{aligned} \tag{4.34}$$

Some explainations of each term in the Eq. (4.34) are in order. In the first place, as mentioned in the previous section, the Eq. (4.34) is the analog of the usual QBE for the quasi-particle distribution function $\delta f(\mathbf{p},\mathbf{q},\Omega)$, thus the structures of the QBEs in both cases are similar. The first term on the left hand side of the equation corresponds to the free fermions. The second term on the left hand side corresponds to the self-energy correction which renormalizes the mass of the fermions. The third term on the left hand side can be regarded as the contribution from the generalized Landau-interaction function which can be defined as

$$F(\theta_{\mathbf{p}'\mathbf{q}}-\theta_{\mathbf{pq}},\omega'-\omega)=v_F^2 \ \mathrm{Re}\ D_{11}(k_F|\theta_{\mathbf{p}'\mathbf{q}}-\theta_{\mathbf{pq}}|,\omega'-\omega)\ . \tag{4.35}$$

Note that this generalized Landau-interaction function contains the frequency dependence as well as the usual angular dependence. This is due to the fact that the gauge interaction is retarded in time and it is also one of the major differences between the fermion-gauge-field system and the usual Fermi liquid. The right hand side of the equation is nothing but the collision integral $I_{\text{collision}}$ and is given by the Fermi-golden-rule. Thus, Eq. (4.34) can be written as

$$\begin{aligned}
&[\ \Omega - v_F q \cos\theta_{\mathbf{pq}}\ ]\ \delta f(\theta_{\mathbf{pq}},\omega)\\
&-[\ \mathrm{Re}\ \Sigma_0^R(\omega+\Omega/2)-\mathrm{Re}\ \Sigma_0^R(\omega-\Omega/2)\ ]\ \delta f(\theta_{\mathbf{pq}},\omega)\\
&+N(0)\int\frac{d\theta_{\mathbf{p}'\mathbf{q}}}{2\pi}\int d\omega'\ F(\theta_{\mathbf{p}'\mathbf{q}}-\theta_{\mathbf{pq}},\omega'-\omega)\\
&\quad\times[\ f_0(\omega+\Omega/2)-f_0(\omega-\Omega/2)\ ]\ \delta f(\theta_{\mathbf{p}'\mathbf{q}},\omega')\\
&=I_{\text{collision}}\ .
\end{aligned} \tag{4.36}$$

After taking the integral $\int d\omega/2\pi$ on both sides of Eq. (4.34), it can be seen that one cannot write the QBE only in terms of $u(\theta_{\mathbf{pq}},\mathbf{q},\Omega)=\int d\omega/2\pi\ \delta f(\theta_{\mathbf{pq}},\omega;\mathbf{q},\Omega)$ which is the generalized Fermi surface displacement. That is, the QBE becomes

$$\begin{aligned}
&[\ \Omega - v_F q \cos\theta_{\mathbf{pq}}\ ]\ u(\theta_{\mathbf{pq}},\mathbf{q},\Omega)\\
&-N(0)\int\frac{d\theta_{\mathbf{p}'\mathbf{q}}}{2\pi}\int d\omega\int d\omega'\ v_F^2\ \mathrm{Re}\ D_{11}(k_F|\theta_{\mathbf{p}'\mathbf{q}}-\theta_{\mathbf{pq}}|,\omega'-\omega)\\
&\quad\times[\ f_0(\omega'+\Omega/2)-f_0(\omega'-\Omega/2)\ ]\ (\delta f(\theta_{\mathbf{pq}},\omega)-\delta f(\theta_{\mathbf{p}'\mathbf{q}},\omega))\\
&=N(0)\int d\theta_{\mathbf{p}'\mathbf{q}}\int_0^\infty\frac{d\nu}{\pi}\int d\omega\int d\omega'\ v_F^2\ \mathrm{Im}\ D_{11}(k_F|\theta_{\mathbf{p}'\mathbf{q}}-\theta_{\mathbf{pq}}|,\nu)\\
&\quad\times[\ \delta(\omega'-\omega+\nu)\ (1-f_0(\omega')+n_0(\nu))+\delta(\omega'-\omega-\nu)\ (f_0(\omega')+n_0(\nu))\ ]\\
&\quad\times(\delta f(\theta_{\mathbf{pq}},\omega)-\delta f(\theta_{\mathbf{p}'\mathbf{q}},\omega))\ .
\end{aligned} \tag{4.37}$$

In the presence of the external potential $U(\mathrm{q},\Omega)$, one should add an additional term $v_F q\ \cos\theta_{\mathbf{pq}}\ U(\mathbf{q},\Omega)$ in the left hand side of Eq. (4.37), which requires a careful derivation. Note that the contributions from the self-energy and the generalized Landau-interaction-function are combined in the left hand side of the QBE. Even though the above equation is already useful, it is worthwile to transform this equation to the more familar one. In the next section, we provide the approximate QBE for $u(\theta_{\mathbf{pq}},\mathbf{q},\Omega)$ which is more useful to understand the low energy excitations of the system.

4.4 Quantum Boltzmann Equation for the Generalized Fermi Surface Displacement

In order to transform the QBE given by Eq. (4.36) or Eq. (4.37) to a more familiar form, it is necessary to simplify the generalized Landau-interaction-function $F(\theta, \omega) = v_F^2$ Re $D_{11}(k_F|\theta|, \omega)$. Note that

$$\text{Re } D_{11}(k_F|\theta|, \omega) = \frac{(\chi/\gamma^2)\ k_F^{2+\eta}\ |\theta|^{2+\eta}}{\omega^2 + (\chi/\gamma)^2\ k_F^{2+2\eta}\ |\theta|^{2+2\eta}} \ . \tag{4.38}$$

It can be checked from Eq. (4.37) that $\delta f(\theta_{\mathbf{pq}}, \omega; \mathbf{q}, \Omega)$ is finite only when $|\omega| \leq \Omega$ at zero temperature. Therefore, the frequency ω in Re $D_{11}(k_F|\theta|, \omega)$ is cutoff by Ω. In this case, one can introduce the Ω dependent cutoff $\theta_c \approx \frac{1}{k_F}\left(\frac{\gamma\Omega}{\chi}\right)^{\frac{1}{1+\eta}}$ in the angle variable and approximate $F(\theta, \omega)$ by the following $F_{\text{Landau}}(\theta)$.

$$F_{\text{Landau}}(\theta) = \begin{cases} F(\theta, \omega = 0)\ , & \text{if } |\theta| > \theta_c\ ; \\ F(\theta = \theta_c, \omega = 0)\ , & \text{otherwise}\ , \end{cases} \tag{4.39}$$

where

$$F(\theta, \omega = 0) = \frac{v_F^2}{\chi k_F^\eta}\,\frac{1}{|\theta|^\eta} \ . \tag{4.40}$$

Using this approximation and $f_0(\omega) = \Theta(-\omega)$ at zero temperature, the QBE given by Eq. (4.37) at zero temperature can be transformed into (the finite temperature case is discussed in the appendix B)

$$\begin{aligned}
&[\ \Omega - v_F q \cos\theta_{\mathbf{pq}}\ ]\ u(\theta_{\mathbf{pq}}, \mathbf{q}, \Omega) \\
&+ \Omega\ N(0) \int \frac{d\theta_{\mathbf{p'q}}}{2\pi}\ F_{\text{Landau}}(\theta_{\mathbf{p'q}} - \theta_{\mathbf{pq}})\ (\ u(\theta_{\mathbf{pq}}, \mathbf{q}, \Omega) - u(\theta_{\mathbf{p'q}}, \mathbf{q}, \Omega)\ ) \\
&= N(0) \int d\theta_{\mathbf{p'q}} \int_0^\infty \frac{d\nu}{\pi} \int d\omega \int d\omega'\ v_F^2\ \text{Im}\ D_{11}(k_F|\theta_{\mathbf{p'q}} - \theta_{\mathbf{pq}}|, \nu) \\
&\quad \times [\ \delta(\omega' - \omega + \nu)\ (1 - f_0(\omega')) + \delta(\omega' - \omega - \nu)\ f_0(\omega')\ ] \\
&\quad \times (\delta f(\theta_{\mathbf{pq}}, \omega) - \delta f(\theta_{\mathbf{p'q}}, \omega)) \ .
\end{aligned} \tag{4.41}$$

Note that $\Omega\ N(0) \int d\theta_{\mathbf{p'q}}/2\pi\ F_{\text{Landau}}(\theta_{\mathbf{p'q}} - \theta_{\mathbf{pq}}) \propto \Omega^{\frac{2}{1+\eta}}$ $(1 < \eta \leq 2)$ or $\Omega \ln \Omega$ $(\eta = 1)$ corresponds to the contribution from the real part of the retarded self-energy. On the other hand, $\Omega\ N(0) \int d\theta_{\mathbf{p'q}}/2\pi\ F_{\text{Landau}}(\theta_{\mathbf{p'q}} - \theta_{\mathbf{pq}})\ u(\theta_{\mathbf{p'q}}, \mathbf{q}, \Omega)$ represents the Landau-interaction part.

For smooth fluctuations of the generalized Fermi surface displacement, $u(\theta, \mathbf{q}, \Omega)$ is a slowly varying function of θ so that there is a forward-scattering cancellation between the self-energy part and the Landau-interaction part. Therefore, for smooth fluctuations, the singular behavior of the self-energy does not appear in the dynamics of the generalized Fermi surface displacement. One the other hand, for rough fluctuations, $u(\theta, \mathbf{q}, \Omega)$ is a fastly varying function. In this case, the Landau-interaction part becomes very small and the self-energy part dominates. Thus, for rough fluctuations, the dynamics of the generailzed Fermi surface displacement should show the singular behavior of the self-energy. From these results, one can expect that the smooth and the rough fluctuations provide very different physical pictures for the elementary excitations of the system.

One can make this observation more concrete by looking at the QBE in angular momentum l (which is the conjugate variable of θ) space. By the following Fourier expansion,

$$u(\theta, \mathbf{q}, \Omega) = \sum_l e^{il\theta}\, u_l(\mathbf{q}, \Omega) \text{ and } \delta f(\theta, \omega; \mathbf{q}, \Omega) = \sum_l e^{il\theta}\, \delta f_l(\omega; \mathbf{q}, \Omega) , \tag{4.42}$$

one can get

$$\begin{aligned} &\Omega\, u_l(\mathbf{q}, \Omega) - \frac{v_F q}{2} \left[\, u_{l+1}(\mathbf{q}, \Omega) + u_{l-1}(\mathbf{q}, \Omega)\, \right] \\ &+ \Omega\, N(0) \int \frac{d\theta}{2\pi}\, F_{\text{Landau}}(\theta)\, (1 - \cos\, (l\theta))\, u_l(\mathbf{q}, \Omega) \\ &= N(0) \int d\theta \int_0^\infty \frac{d\nu}{\pi} \int d\omega \int d\omega'\, v_F^2\, \text{Im}\, D_{11}(k_F|\theta|, \nu)\, (1 - \cos\, (l\theta)) \\ &\times [\, \delta(\omega' - \omega + \nu)\, (1 - f_0(\omega')) + \delta(\omega' - \omega - \nu)\, f_0(\omega')\,]\, \delta f_l(\omega; \mathbf{q}, \Omega)\, . \end{aligned} \tag{4.43}$$

Note that, in the $1 - \cos\, (l\theta)$ factor inside the integral on the left hand side of the QBE given by Eq. (4.43), 1 comes from the self-energy part and $\cos\, (l\theta)$ comes from the Landau-interaction part. For $l < l_c \approx 1/\theta_c \propto \Omega^{-\frac{1}{1+\eta}}$, $1 - \cos\, (l\theta) \approx l^2\theta^2/2$ and the additional θ^2 dependence makes the angle integral less sigular because typical θ is of the order of $\Omega^{\frac{1}{1+\eta}}$. Due to this cancellation for the small angle (forward) scattering, the correction from the self-energy part and the Landau-interaction part becomes of the order of $\Omega^{\frac{4}{1+\eta}}$ so that it does not cause any sigular correction. Note that a similar type of cancellation occurs in the collision integral. Therefore, for the small angular momentum modes $l < l_c$, the system behaves like the usual Fermi liquid. For $l > l_c$, the $\cos\, (l\theta)$ factor becomes highly oscillating as a function of θ so that the Landau-interaction part becomes very small. As a result, the self-energy part dominates and the dispersion relation for the dynamics of the generalized Fermi surface displacement is changed from $\Omega = v_F q$ to $\Omega \propto q^{\frac{1+\eta}{2}}$ $(1 < \eta \leq 2)$ or $\Omega \propto q/|\ln q|$ $(\eta = 1)$. Also, a similar thing happens in the collision integral, *i.e.*, the $\cos\, (l\theta)$ factor does not contribute and the remaining contribution shows the singular behavior of the imaginary part of the self-energy so that the collision integral cannot be ignored for $1 < \eta \leq 2$ and can be *marginally* ignored for $\eta = 1$.

Using the above results, one can understand the density-density and the current-current correlation functions which show no anomalous behavior in the long wave length and the low frequency limits [13, 14]. From the QBE, one can evaluate these correlation functions by taking the angular average of the density or the current disturbance due to the external potential and calculating the linear response. As a result, in these correlation functions, the small angular momentum modes are dominating so that the results do not show any singular behavior. From these results, one can also expect that two different behaviors of the small ($l < l_c$) and the large ($l > l_c$) angular momentum modes may show up even in the presence of the finite effective magnetic field ΔB and the large angular momentum modes may be responsible for the singular energy gap of the system [6, 15, 16], which is the subject of the next section.

4.5 Quantum Boltzmann Equation in the Presence of Effective Magnetic Field and the Energy Gap

We follow Hänsch and Mahan [39] to derive the QBE in the presence of the finite effective magnetic field ΔB. The only difference between the case of $\Delta B \neq 0$ and that of $\Delta B = 0$ is that the external vector potential $\Delta\mathbf{A} = -\frac{1}{2}\mathbf{r} \times \Delta\mathbf{B}$ enters to the knietic energy in the equation of motion of the one-particle Green's function [39]. The same procedure used in the case of $\Delta B = 0$ can be imployed to derive the QBE from the equation of motion of the one-particle Green's function. The resulting equation can be transformed to a convenient form by a change of variables given by

$$\mathbf{P} = \mathbf{p} - \Delta\mathbf{A} = \mathbf{p} + \frac{1}{2}\mathbf{r} \times \Delta\mathbf{B} \tag{4.44}$$

so that one can construct the QBE for $G^<(\mathbf{P}, \omega; \mathbf{q}, \Omega)$ which is now a function of $\mathbf{P}$ [39]. As a result, the change we have to make for the case of $\Delta B \neq 0$ (compared to the case of $\Delta B = 0$ given by Eq. (4.20)) is that all the momentum variables should be changed from $\mathbf{p}$ to $\mathbf{P}$ and the following additional terms should be added to Eq. (4.20) [39].

$$\begin{aligned}
&\frac{\mathbf{P}}{m} \cdot \Delta\mathbf{B} \times \frac{\partial}{\partial\mathbf{P}}\delta G^<(\mathbf{P}, \omega; \mathbf{q}, \Omega) + \frac{\partial}{\partial\mathbf{P}}\delta(\mathrm{Re}\ \Sigma^R(\mathbf{P}, \omega; \mathbf{q}, \Omega)) \cdot \Delta\mathbf{B} \times \frac{\partial}{\partial\mathbf{P}} G_0^<(\mathbf{P}, \omega) \\
&-\Delta\mathbf{B} \cdot \frac{\partial}{\partial\mathbf{P}}\delta\Sigma^<(\mathbf{P}, \omega; \mathbf{q}, \Omega) \\
&+\Delta\mathbf{B} \cdot \frac{\partial}{\partial\mathbf{P}}\Sigma_0^<(\mathbf{P}, \omega; \mathbf{q}, \Omega) \times \frac{\partial}{\partial\mathbf{P}}\delta(\mathrm{Re}\ G^R(\mathbf{P}, \omega; \mathbf{q}, \Omega)) \ .
\end{aligned} \tag{4.45}$$

Since the self-energy does not depend on the momentum $\mathbf{P}$ in the fermion-gauge-field system, the only term which contributes to the QBE is

$$\frac{\mathbf{P}}{m} \cdot \Delta\mathbf{B} \times \frac{\partial}{\partial\mathbf{P}}\delta G^<(\mathbf{P}, \omega; \mathbf{q}, \Omega) \ . \tag{4.46}$$

In principle, the self-enegy and the Green's function in the QBE also depend on the effective magnetic field ΔB. In the semiclassical approximation for very small ΔB, we ignore this type of ΔB dependence and, instead of that, we introduce a low energy cutoff E_g in the frequency integrals, which is the energy gap of the system. Then, after the integration $\int d\xi_{\mathbf{P}}/2\pi$, the equation becomes that of Eq. (4.20) with a low energy cutoff E_g and it also contains an additional term given by

$$\Delta\omega_c \frac{\partial}{\partial\theta_{\mathbf{Pq}}} \ \delta f(\theta_{\mathbf{Pq}}, \omega; \mathbf{q}, \Omega) \ , \tag{4.47}$$

where $\Delta\omega_c = \Delta B/m$. After $\int d\omega/2\pi$, the QBE for a generalized Fermi surface displacement can be written as

$$
\begin{aligned}
& [\,\Omega - v_F q \cos\theta_{\mathbf{Pq}}\,]\; u(\theta_{\mathbf{Pq}},\mathbf{q},\Omega) - i\Delta\omega_c \frac{\partial}{\partial\theta_{\mathbf{Pq}}}\; u(\theta_{\mathbf{Pq}},\mathbf{q},\Omega) \\
& + \Omega\; N(0) \int \frac{d\theta_{\mathbf{P'q}}}{2\pi}\; F_{\mathrm{Landau}}(\theta_{\mathbf{P'q}} - \theta_{\mathbf{Pq}})\;(\; u(\theta_{\mathbf{Pq}},\mathbf{q},\Omega) - u(\theta_{\mathbf{P'q}},\mathbf{q},\Omega)\;) \\
& = N(0) \int d\theta_{\mathbf{P'q}} \int_0^\infty \frac{d\nu}{\pi} \int d\omega \int d\omega'\; v_F^2\; \mathrm{Im}\; D_{11}(k_F|\theta_{\mathbf{P'q}} - \theta_{\mathbf{Pq}}|,\nu) \\
& \quad \times[\;\delta(\omega' - \omega + \nu)\;(1 - f_0(\omega')) + \delta(\omega' - \omega - \nu)\; f_0(\omega')\;] \\
& \quad \times(\delta f(\theta_{\mathbf{Pq}},\omega) - \delta f(\theta_{\mathbf{P'q}},\omega))\;,
\end{aligned} \tag{4.48}
$$

where a low energy cutoff E_{g} is introduced in the frequency integrals. In particular, the angle cutoff θ_c in $F_{\mathrm{Landau}}(\theta)$ should be changed from $\theta_c \approx \frac{1}{k_F}\left(\frac{\gamma\Omega}{\chi}\right)^{\frac{1}{1+\eta}}$ $(\Delta B = 0)$ to $\theta_c \approx \frac{1}{k_F}\left(\frac{\gamma E_{\mathrm{g}}}{\chi}\right)^{\frac{1}{1+\eta}}$ $(\Delta B \neq 0)$ in the low frequency Ω limit.

Now similar interpretations can be made as the case of $\Delta B = 0$. For the smooth fluctuations ($l < l_c \approx 1/\theta_c$), there is a cancellation between the self-energy and the Landau-interaction parts. As a result, we have a term which is the order of $\Omega\; E_{\mathrm{g}}^{\frac{3-\eta}{1+\eta}}$ which can be ignored compared to Ω because E_{g} is very small near $\nu = 1/2$ or $\Delta B = 0$. Also, a similar thing happens in the collision integral. Therefore, the QBE for the smooth fluctuations can be written as

$$
[\,\Omega - v_F q \cos\theta_{\mathbf{Pq}}\,]\; u(\theta_{\mathbf{Pq}},\mathbf{q},\Omega) - i\Delta\omega_c \frac{\partial}{\partial\theta_{\mathbf{Pq}}}\; u(\theta_{\mathbf{Pq}},\mathbf{q},\Omega) \approx 0\;. \tag{4.49}
$$

On the other hand, for the rough fluctuations ($l > l_c$), the self-energy part dominates and we have a contribution which is of the order $\Omega\; E_{\mathrm{g}}^{-\frac{\eta-1}{\eta+1}}$ $(1 < \eta \leq 2)$ or $\Omega\;|\ln\, E_{\mathrm{g}}|$ $(\eta = 1)$. Ignoring Ω term compared to $\Omega\; E_{\mathrm{g}}^{-\frac{\eta-1}{\eta+1}}$ $(1 < \eta \leq 2)$ or $\Omega\;|\ln\, E_{\mathrm{g}}|$ $(\eta = 1)$ and multifliying the factor $E_{\mathrm{g}}^{\frac{\eta-1}{\eta+1}}$ $(1 < \eta \leq 2)$ or $1/|\ln\, E_{\mathrm{g}}|$ $(\eta = 1)$ on both sides of the equation, we get

$$
[\,\Omega - v_F^* q \cos\theta_{\mathbf{Pq}}\,]\; u(\theta_{\mathbf{Pq}},\mathbf{q},\Omega) - i\Delta\omega_c^* \frac{\partial}{\partial\theta_{\mathbf{Pq}}}\; u(\theta_{\mathbf{Pq}},\mathbf{q},\Omega) = \text{collision integral}\;, \tag{4.50}
$$

where $v_F^* = k_F/m^*$, $\Delta\omega_c^* = \Delta B/m^*$, and $m^*/m \propto E_{\mathrm{g}}^{-\frac{\eta-1}{\eta+1}}$ $(1 < \eta \leq 2)$ or $|\ln\, E_{\mathrm{g}}|$ $(\eta = 1)$.

Let us consider two different types of wave packets created along the Fermi surface. Note that the revolution of these wave packets is governed by two different frequencies $\Delta\omega_c$ and $\Delta\omega_c^*$. The frequency of revolution of the broad wave packet (see Figure 4-2 (a)) is given by $\Delta\omega_c$ because it mainly consists of small angular momentum modes. On the other hand, if we ignore the collision integral in the QBE, the frequency of revolution of the narrow wave packet (see Figure 4-2 (b)) is given by $\Delta\omega_c^*$ because it mainly contains the large angular momentum modes.

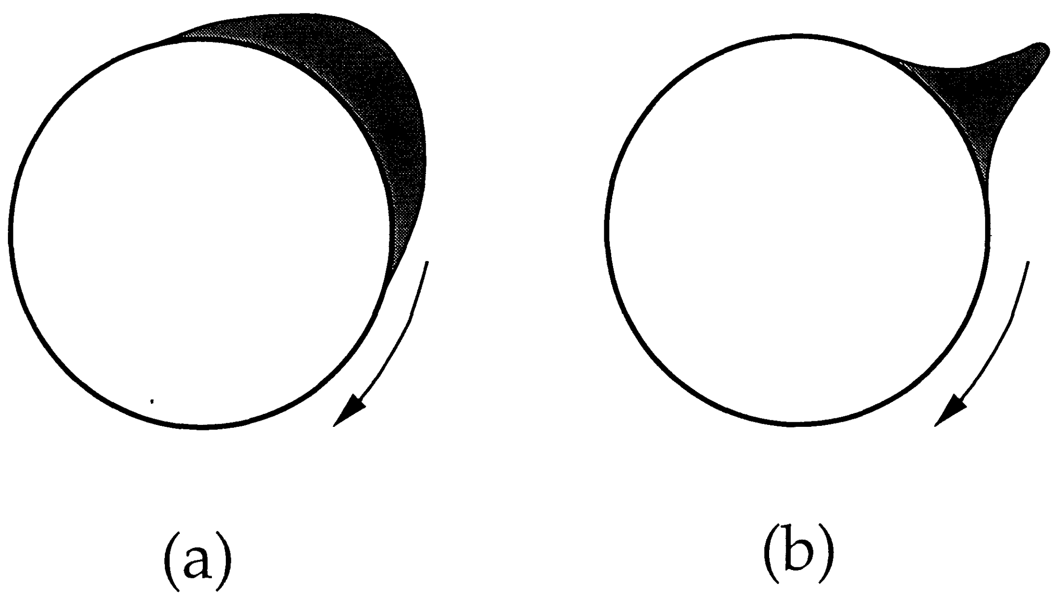


Figure 4-1: A broad wave packet (a) and a narrow wave packet (b) (given by the shaded region) created in the momentum space. The circle is the schematic representation of the Fermi surface, which is actually not so well defined, and the arrow represents the direction of motion of the wave packet.

The energy gap of the system can be obtained by quantizing the motion of revolution and taking the smallest quantized frequency as the energy gap of the system. Therefore, the energy gap of the system is given by $E_{\rm g} = \Delta\omega_c^* \propto \Delta B\, E_{\rm g}^{\frac{\eta-1}{\eta+1}}$ $(1 < \eta \leq 2)$ or $\Delta B/|\ln\, E_{\rm g}|$ $(\eta = 1)$. Solving this self-consistent equation for $E_{\rm g}$, we get

$$E_{\rm g} \propto \begin{cases} |\Delta B|^{\frac{1+\eta}{2}} , & \text{if } 1 < \eta \leq 2 \ ; \\ \frac{|\Delta B|}{|\ln\, \Delta B|} , & \text{if } \eta = 1 \ . \end{cases} \tag{4.51}$$

This result is the same as the self-consistent treatment of HLR [6] and also the perturbative evaluation of the activation energy gap in the finite temperature compressibility [15]. We see that the divergent effective mass shows up in the energy gap $E_{\rm g}$. More detailed discussions of the low lying excitations described by the QBE can be found in the next section.

4.6 Collective Excitations

Let us first study the collective excitations of the system with $\Delta B = 0$ by looking at the QBE given by Eq. (4.43). We ignore the collision integral for the time being and discuss its influence later. In the absence of the collision integral, Eq. (4.43) can be considered as the Schrödinger equation of an equivalent tight binding model in the angular momentum space. It is convenient to rewrite Eq. (4.43) as

$$\begin{aligned} \Omega\, v_l &= \frac{v_F q}{2} \left[\frac{v_{l+1}}{\sqrt{g(l)g(l+1)}} + \frac{v_{l-1}}{\sqrt{g(l)g(l-1)}} \right] , \\ v_l &= \sqrt{g(l)}\, u_l , \end{aligned} \tag{4.52}$$

where

$$g(l,\Omega) = 1 + \, N(0) \int \frac{d\theta}{2\pi}\, F_{\rm Landau}(\theta)\, (1 - \cos\,(l\theta)) \ . \tag{4.53}$$

Eq. (4.52) describes a particle hopping in a 1D lattice with a 'spatial' dependent hopping amplitude $t_l \approx \frac{v_F q}{2g(l)}$. Note that $g(l)$ is of the order one for $l < l_c$ and becomes much larger, $g(l) \propto \Omega^{-\frac{\eta-1}{\eta+1}}$, when $l > l_c$. Due to this type of 'spatial' dependent hopping amplitude, the eigenspectrum of Eq. (4.52) consists of two parts. That is, there is a continuous spectrum near the center of the band and a discrete spectrum in the tail of the band. The descrete spectrum appears above and below the continuous spectrum (See Figure 4-2).

The boundary between these two different spectra is determined from $\Omega = 2t_{l\to\infty} \propto v_F q\, \Omega^{\frac{\eta-1}{\eta+1}}$, which self-consistantly generates a singular dispersion relation $\Omega(\theta) \propto q^{\frac{1+\eta}{2}}$ $(1 < \eta \leq 2)$ or $\Omega(\theta) \propto q/|\ln\, q|$ $(\eta = 1)$. Also, the tail of the band ends at $\Omega(\theta) = 2t_1 \sim v_F q$. One can map this energy spectrum to the diagram for the excitations in the usual $\Omega - q$ plane, which is given by the Figure 4-3.

Note that the continuum states $(l > l_c)$ can be mapped to the particle-hole con-

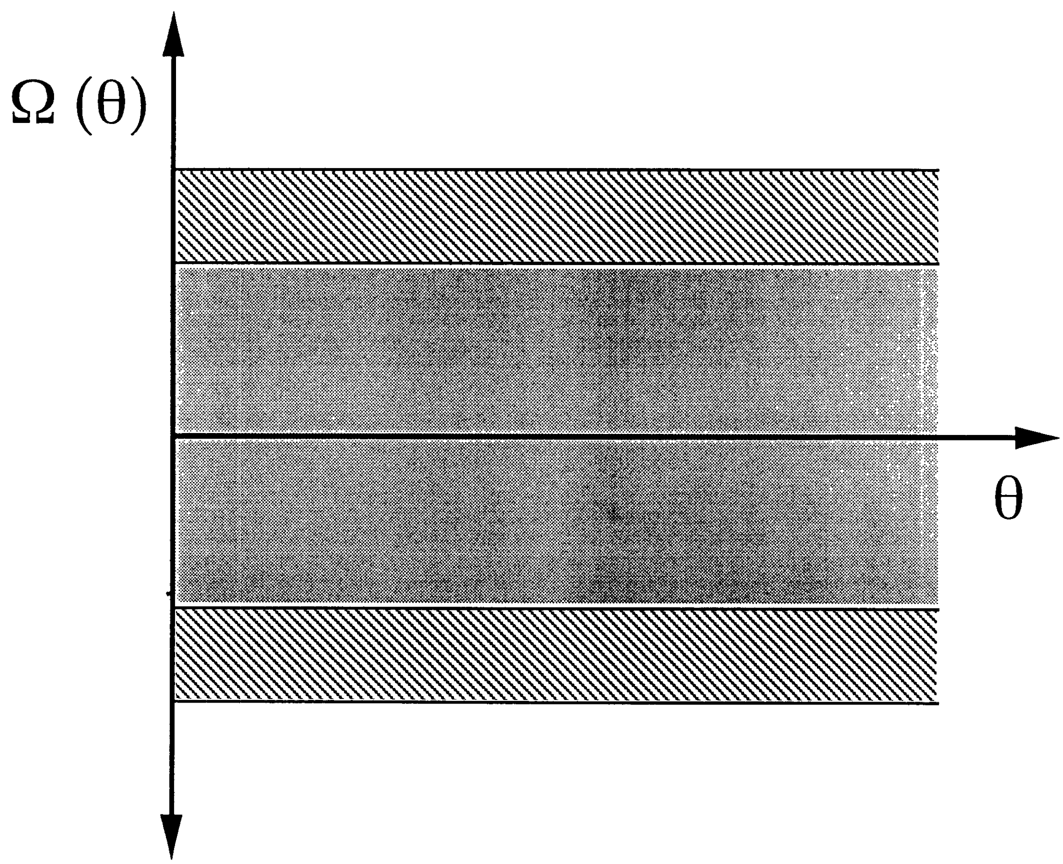


Figure 4-2: The energy band $\Omega(\theta)$ of the tight binding model given by Eq. (4.52) as a function of θ. The shaded region around the center of the band corresponds to the continuum states and the hatched region in the tails of the band corresponds to the bound states.

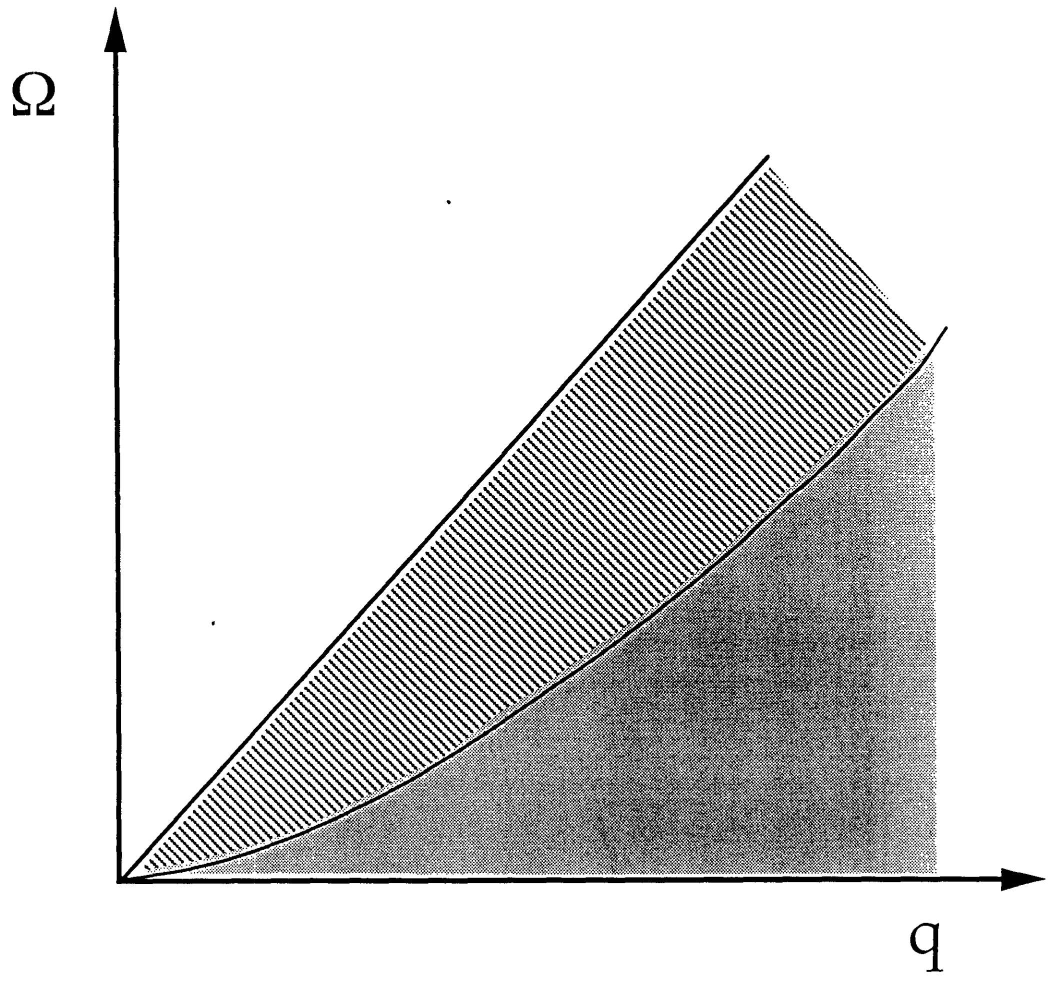


Figure 4-3: The elementary excitations in $\Omega - q$ space in the absence of the collision integral. The shaded region corresponds to the particle-hole continuum and the hatched region corresponds to the collective modes. The boundary is given by the singular dispersion relation $\Omega \propto q^{\frac{1+\eta}{2}}$ for $1 < \eta \leq 2$ and $\Omega \propto q/|\ln q|$ for $\eta = 1$.

tinuum which exist below $\Omega \propto q^{\frac{1+\eta}{2}}$ $(1 < \eta \leq 2)$ or $\Omega \propto q/|\ln q|$ $(\eta = 1)$. On the other hand, the bound states (the discrete spectrum) $(l < l_c)$ can be mapped to the collective modes which exist between $\Omega \propto q^{\frac{1+\eta}{2}}$ $(1 < \eta \leq 2)$, $\Omega \propto q/|\ln q|$ $(\eta = 1)$ and $\Omega \sim v_F q$. However, the distinction between these two different elementary excitations is obscured by the presence of the collision integral which provides the life time for the excitations. In particular, since $g(l, \Omega)$ does not provide a sharp boundary between $l > l_c$ and $l < l_c$, one expects a crossover from the particle-hole excitations to the collective modes even in the absence of the collision integral.

Now let us consider the case of $\Delta B \neq 0$ (*i.e.*, away from $\nu = 1/2$ state). In this case, Eq. (4.52) becomes (see also Eq. (4.48))

$$\Omega\ v_l = \frac{l\Delta\omega_c}{g(l)} v_l + \frac{v_F q}{2} \left[\frac{v_{l+1}}{\sqrt{g(l)g(l+1)}} + \frac{v_{l-1}}{\sqrt{g(l)g(l-1)}} \right] . \tag{4.54}$$

When $g(l) = 1$, one can write the solution of Eq.(63) (or Eq.(57)) as

$$u(\theta_{\mathbf{Pq}}, \mathbf{q}, t) \propto e^{in\theta_{\mathbf{Pq}} - i\Omega t}\ e^{-i\frac{v_F q}{\Delta\omega_c} \sin\theta_{\mathbf{Pq}}} \tag{4.55}$$

with $\Omega = n\Delta\omega_c$. Thus, we recover the well known spectrum of degenerate Landau levels for free fermions.

When $g(l) \neq$ const., it is difficult to calculate the spectrum of Eq. (4.54). However, using $g(l) = g(-l)$, we can show that the spectrum of Eq. (4.54) is symmetric about $\Omega = 0$, and $\Omega = 0$ is always an eigenvalue of Eq. (4.54). Also, for non-zero $\Delta\omega_c$, the spectrum is always discrete.

Note that, for small $q \ll l_c\Delta\omega_c/v_F$, $u(\theta_{\mathbf{Pq}}, \mathbf{q}, t)$ corresponds to a smooth fluctuation of the Fermi surface. While, for large $q \gg l_c\Delta\omega_c/v_F$, even the smooth parts of $u(\theta_{\mathbf{Pq}}, \mathbf{q}, t)$, around $\theta_{\mathbf{Pq}} = \pm\pi/2$, correspond to a rough fluctuation, hence the whole function $u(\theta_{\mathbf{Pq}}, \mathbf{q}, t)$ corresponds to a rough fluctuation. Thus, we expect that the small q modes and the large q modes have very different dynamics. The small q modes should be controled by the finite effective mass and the large q modes, the divergent mass.

To understand the behavior of the modes in more detail, in the following, we present a semiclassical calculation. The main result that we obtain is the Eq. (4.69). The dispersion of the lowest lying mode (for $q > \Delta\omega_c/v_F$) has a scaling form $\omega_{\text{cyc}}(q) \propto (\Delta\omega_c)^{\frac{1+\eta}{2}} f(q/q_c)$ with $f(\infty) =$ const. and $f(x \ll 1) \propto x^{1-\eta}$. The crossover momentum $q_c \propto \sqrt{\Delta\omega_c}$.

When $qv_F \ll \Delta\omega_c$ the spectrum can be calculated exactly and is given by

$$\Omega = \frac{l\Delta\omega_c}{g(l)} . \tag{4.56}$$

To obtain the spectrum for $qv_F > \Delta\omega_c$ we will use a semiclassical approach. Note that $(\theta_{\mathbf{Pq}}, l)$ is a canonical coordinate and momentum pair. The classical Hamiltonian that corresponds to the quantum system Eq. (4.54) can be found to be

$$H(\theta_{\mathbf{Pq}}, l) = \frac{l\Delta\omega_c}{g(l)} + \frac{v_F q}{g(l)}\cos(\theta_{\mathbf{Pq}}) \ . \tag{4.57}$$

Assuming $g(l)$ is a slowly varying function of l, one arrives at the following simple classical equations of motion

$$\dot{\theta}_{\mathbf{Pq}} = \frac{\Delta\omega_c}{g(l)}, \qquad \dot{l} = \frac{v_F q}{g(l)}\sin(\theta_{\mathbf{Pq}}) \ . \tag{4.58}$$

From this equation, one can easily show that

$$l = -\frac{v_F q}{\Delta\omega_c}\cos(\theta_{\mathbf{Pq}}) + l_0 \ , \tag{4.59}$$

where l_0 is a constant. Note that Eq. (4.59) with $l_0 = 0$ is an exact solution for the classical system Eq. (4.57), which describes a motion with zero energy. Now the first equation in Eq. (4.58) can be simplified as

$$\dot{\theta}_{\mathbf{Pq}} = \frac{\Delta\omega_c}{g(-\frac{v_F q}{\Delta\omega_c}\cos(\theta_{\mathbf{Pq}}) + l_0)} \ , \tag{4.60}$$

which describes a periodic motion. The angular frequency of the periodic motion is given by

$$\omega = \frac{2\pi\Delta\omega_c}{\int_0^{2\pi} g(-\frac{v_F q}{\Delta\omega_c}\cos(\theta_{\mathbf{Pq}}) + l_0)\ d\theta_{\mathbf{Pq}}} \ . \tag{4.61}$$

The above classical frequency ω has a quantum interpretation. It is the gap between neighboring energy levels, of which the energy is close to the classical energy associated with the classical motion described by Eq. (4.59). In particular, the cyclcotron frequency ω_{cyc} is given by the gap between the $\Omega = 0$ level and the first $\Omega > 0$ level. Therefore

$$\omega_{\text{cyc}} = \frac{2\pi\Delta\omega_c}{\int_0^{2\pi} g(-\frac{v_F q}{\Delta\omega_c}\cos(\theta_{\mathbf{Pq}}) + 1)\ d\theta_{\mathbf{Pq}}} \ . \tag{4.62}$$

Here we have chosen $l_0 = 1$ (instead of $l_0 = 0$) so that Eq. (4.62) repreduces the exact result Eq. (4.56) when $q = 0$. Note that $g(l)$ also depends on frequency Ω and we should set $\Omega = \omega_{\text{cyc}}$ in the function $g(l)$. Thus, the cylcotron frequency should be self-consistantly determined from Eq. (4.62).

We would like to remark that when $q \gg \Delta\omega_c/v_F$, the classical frequency in Eq. (4.61) is a smooth function of l_0, hence a smooth function of the energy. This means that the gap between the neighboring energy levels is also a smooth function of the energy of the levels. The validity of the semiclassical approach requires that the gap between neighboring energy levels is almost a constant in the neighborhood of interested energies. Thus the above behavior of the classical frequency indicates that the semiclassical approach is at least self-consistant.

To analyze the behavior of $\omega_{\rm cyc}$, we first make an approximation for Eq. (4.62) as

$$\omega_{\rm cyc} = \frac{\Delta\omega_c}{g(\lambda\frac{v_F q}{\Delta\omega_c}+1)} \ , \tag{4.63}$$

where λ is a non-zero constant between 0 and 1. We see that $\omega_{\rm cyc}(q)$ has a sharp dependence on q around $q \sim \Delta\omega_c/v_F$. The smaller the $\Delta\omega_c$ the sharper the q dependence. This sharp dependence is not due to the singular gauge interaction, but merely a consequence of the fact that $g(1) \neq g(2) \neq \cdots$.

As q increases, $g(\lambda\frac{v_F q}{\Delta\omega_c}+1)$ becomes larger and larger, thus we expect that $\omega_{\rm cyc}(q)$ decreases. When q exceeds a crossover value q_c, $g(\lambda\frac{v_F q}{\Delta\omega_c}+1)$ saturates at a very large value and $\omega_{\rm cyc}(q)$ is drastically reduced. This phenomena is a result of the singular gauge interaction. The crossover momentum q_c is determined from

$$\frac{v_F q_c}{\Delta\omega_c} = l_c = k_F\left(\frac{\chi}{\gamma\omega_{\rm cyc}(q\to\infty)}\right)^{\frac{1}{1+\eta}} \ , \tag{4.64}$$

and

$$\begin{aligned}
\omega_{\rm cyc}(q\to\infty) &= \frac{\Delta\omega_c}{C(\eta)(\omega_{\rm cyc}(q\to\infty))^{\frac{1-\eta}{1+\eta}}} \text{ for } 1<\eta\leq 2, \\
\omega_{\rm cyc}(q\to\infty) &= \frac{\Delta\omega_c}{C(\eta=1)|\ln\,\omega_{\rm cyc}(q\to\infty)|} \text{ for } \eta=1 \ ,
\end{aligned} \tag{4.65}$$

where

$$C(\eta) = \frac{v_F\,\cos\left[\frac{\pi}{2}\left(\frac{\eta-1}{\eta+1}\right)\right]}{2\pi(1+\eta)\,\sin\left(\frac{2\pi}{1+\eta}\right)\gamma^{\frac{\eta-1}{\eta+1}}\chi^{\frac{2}{1+\eta}}} \tag{4.66}$$

for $1<\eta\leq 2$ and $C(\eta=1)=\frac{v_F}{2\pi^2\chi}$ for $\eta=1$. We find

$$\begin{aligned}
q_c &= B(\eta)\sqrt{\Delta\omega_c} \quad \text{for } 1<\eta\leq 2 \ , \\
q_c &= B(\eta=1)\sqrt{\Delta\omega_c|\ln\,\Delta\omega_c|} \quad \text{for } \eta=1 \ ,
\end{aligned} \tag{4.67}$$

where $B(\eta)=m(\chi/\gamma)^{\frac{1}{1+\eta}}\sqrt{C(\eta)}$. When $q\gg q_c$, the cyclotron frequency saturates at the following values.

$$\begin{aligned}
\omega_{\rm cyc}(q\to\infty) &= (\Delta\omega_c/C(\eta))^{\frac{1+\eta}{2}} \quad \text{for } 1<\eta\leq 2 \ , \\
\omega_{\rm cyc}(q\to\infty) &= \frac{\Delta\omega_c/C(\eta=1)}{|\ln(\Delta\omega_c/C(\eta=1))|} \quad \text{for } \eta=1 \ .
\end{aligned} \tag{4.68}$$

When $v_F q/\Delta\omega_c \gg 1$, the cyclotron freqency is expected to have the following scaling form:

$$\begin{aligned}
\omega_{\rm cyc}(q) &\propto (\Delta\omega_c)^{\frac{1+\eta}{2}} f(q/q_c) \ , \\
f(\infty) &= \text{const.} \quad \text{and} \quad f(x\ll 1)\propto x^{1-\eta} \ ,
\end{aligned} \tag{4.69}$$

where $f(\infty)$ is determined from $\omega_{\rm cyc}(q \to \infty) \propto (\Delta\omega_c)^{\frac{1+\eta}{2}}$ and $f(x \ll 1)$ can be obtained from the condition that $\omega_{\rm cyc}(q) = \Delta\omega_c$ for $q \sim \Delta\omega_c/v_F$. Note that the divergence of $f(x)$ for small x should be cutoff when $x \sim \Delta\omega_c/v_F q_c$. As a result, the cyclotron spectrum of the system looks like the one given by Figure 4-4.

The smaller gap for $q > q_c$ corresponds to a divergent effective mass $m^* \propto (\Delta\omega_c)^{\frac{1-\eta}{1+\eta}}$, while the larger gap near $q = 0$ can be viewed as a cyclotron freqency derived from a finite effective mass. The thermal activation gap measured through the longitudinal conductance is given by the smaller gap at large wave vectors $q > q_c$. However the cyclotron frequency measured through the cyclotron resonance for the uniform electric field should be given by the larger gap.

The above discussion of the cyclotron frequency is for the toy model, where only the transverse gauge field fluctuations are included. One may wonder whether the same picture also applies to the real $\nu = 1/2$ state. In the real $\nu = 1/2$ state, the lowest lying plasma modes correspond to the intra-Landau-level excitations, of which energy is much less than the inter-Landau-level gap ω_c. In the $q \to 0$ limit, such modes decouple from the center of mass motion. This means that the $u_{\pm 1}$ components (which correspond to the dipolar distortions of the Fermi surface) of the eigenmodes must disappear in the $q \to 0$ limit as far as the lowest lying modes (intra-Landau-level modes) are concerned. The mode that contains $u_{\pm 1}$ components should have the big inter-Landau-level gap in the $q \to 0$ limit in order to satisfy the Kohn's theorem. Examining our solution for the eigenmodes in the $q \to 0$ limit, we find that the lowest eigenmodes are given by $u_l \propto \delta_{\pm 1, l}$. Therefore, according to the above consideration, we cannot identify the lowest lying modes in the toy model with the lowest lying intra-Landau level plasma modes in the real model. However, this problem can be fixed following the procedure introduced in Ref.[37]. That is, we may introduce an additional non-divergent Landau-Fermi-liquid parameter ΔF_1 which modifies only the value of $g(\pm 1)$. We may fine-tune the value of ΔF_1 such that the $l = \pm 1$ modes in Eq. (4.56) will have the big inter-Landau-level gap $\Omega = \frac{\Delta\omega_c}{g(\pm 1)} = \omega_c$. In this case the $l = \pm 2$ modes become the lowest lying modes in the $q \to 0$ limit. Such modes correspond to the quadradpolar distortions of the Fermi surface and decouple from the center of mass motion. The above correction only affects the energy of the lowest lying modes for the small momenta, $q < \Delta\omega_c/v_F$. With this type of correction, our results for the toy model essentially applies to the $\nu = 1/2$ state. The only change is that the lowest lying modes at small momenta, $q \ll \Delta\omega_c/v_F$, is given by the $l = \pm 2$ modes instead of the $l = \pm 1$ modes. This is because as q decreases below a value of order $\Delta\omega_c/v_F$, the $l = \pm 1$ modes start to have a higher energy than that of the $l = \pm 2$ modes, and the lowest lying modes crossover to the $l = \pm 2$ modes.

In the absence of the singular gauge interaction, according to the picture developed in Ref.[37], one expects that the intra-Landau-level plasma mode near $\nu = 1/2$ has a gap $2\Delta\omega_c$ for $q < \Delta\omega_c/v_F$. The gap is expected to be reduced by the factor 2 when $q > \Delta\omega_c/v_F$. In the presence of the singular gauge interaction, we find that the plasma mode has a gap of order $2\Delta\omega_c$ (since $g(\pm 2) \neq 1$) for $q < \Delta\omega_c/v_F$. However, the gap for the large momenta can be much less than $\Delta\omega_c$. Observing this drastic gap reduction will confirm the presence of the singular gauge interaction.

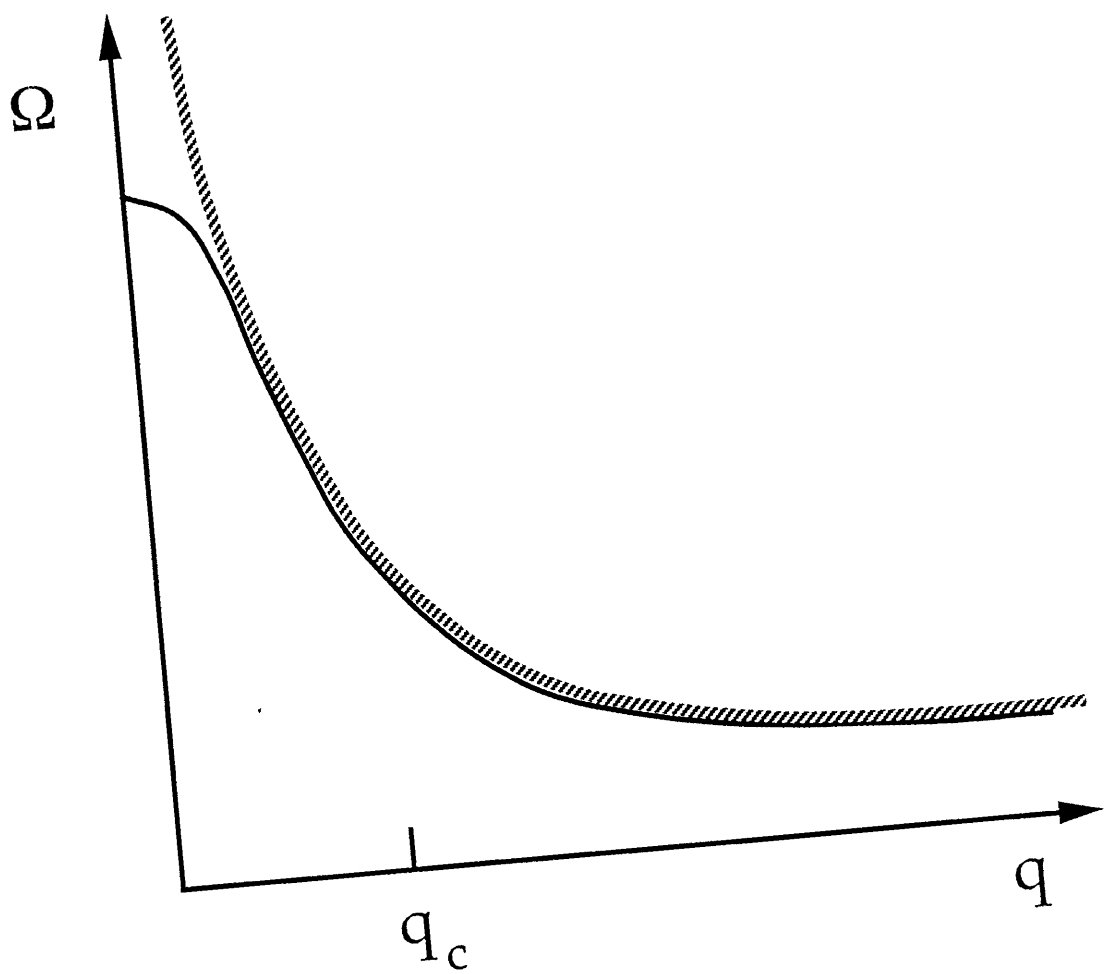


Figure 4-4: The lowest excitation spectrum of the composite fermion system in the presence of the finite effective magnetic field ΔB as a function of the wave vector q (solid line). The dashed line is the scaling curve described in the text. For $q \gg q_c$, the excitation gap becomes smaller and is proportional to $|\Delta B|^{\frac{1+\eta}{2}}$ for $1 < \eta \leq 2$ and $|\Delta B|/|\ln \Delta B|$ for $\eta = 1$. $q_c \propto \sqrt{|\Delta B|}$ for $1 < \eta \leq 2$ and $q_c \propto \sqrt{|\Delta B|\ |\ln \Delta B|}$ for $\eta = 1$.

In the above disscusion, we have ignored the effects of the collision term. The role of collision integral is simply to provide the life time effects on the collective excitations. However, due to the energy conservation, only the collective modes with energy greater than $2\omega_{\rm cyc}(q_{\rm min})$ will have a finite life time. Here $\omega_{\rm cyc}(q_{\rm min})$ is the minimum energy gap of the lowest lying plasma mode and $q_{\rm min}$ is the momentum where the energy takes the minimum value. For large q, the modes above $2\omega_{\rm cyc}(q_{\rm min})$ may have a short life time such that the modes are not well defined.

4.7 Summary, Conclusion, and Implications to Experiments

In this section, we summarize the results and provide the unified picture for the composite fermions interacting with a gauge field. In this paper, we construct a general framework, which is the QBE of the system, to understand the previously known theoretical [6, 13, 14, 15, 16, 17] and experimental [1, 2, 3, 8, 9, 10] results. Since there is no well defined Landau-quasi-particle, we cannot use the usual formulation of the QBE so that we used an alternative formulation which was used by Prange and Kadanoff [32] for the electron-phonon problem. We used the non-equilibrium Green's function technique [32, 33, 34, 35] to derive the QBE of the generalized distribution function $\delta f(\theta_{\mathbf{pq}}, \omega; \mathbf{q}, \Omega)$ for $\Delta B = 0$, and $\delta f(\theta_{\mathbf{Pq}}, \omega; \mathbf{q}, \Omega)$ $(\mathbf{P} = \mathbf{p} - \Delta \mathbf{A})$ for $\Delta B \neq 0$. From this equation, we also derived the QBE for the generalized Fermi surface displacement $u(\theta_{\mathbf{pq}}, \mathbf{q}, \Omega)$ $(\Delta B = 0)$ or $u(\theta_{\mathbf{Pq}}, \mathbf{q}, \Omega)$ $(\Delta B \neq 0)$ which corresponds to the local variation of the chemical potential in momentum space.

For $\Delta B = 0$, the QBE consists of three parts; the self-energy part, the generalized Landau-interaction part, and the collision integral. The Landau-interaction function $F_{\rm Landau}(\theta)$ can be taken as $F_{\rm Landau}(\theta) \propto 1/|\theta|^\eta$ for $\theta > \theta_c \propto \Omega^{\frac{1}{1+\eta}}$ and $1/|\theta_c|^\eta$ for $\theta < \theta_c$. For the smooth fluctuations of the genaralized Fermi surface displacement $(l < l_c \approx 1/\theta_c \propto \Omega^{-\frac{1}{1+\eta}})$, where l (the angular momentum in momentum space) is the conjugate variable of the angle θ, there is a small-angle-(forward)-scattering cancellation between the self-energy part and the Landau-interaction part. Both of the self-energy part and the Landau-interaction part are of the order $\Omega^{\frac{2}{1+\eta}}$ $(1 < \eta \leq 2)$ or $\Omega \ln \Omega$ $(\eta = 1)$. After cancellation, the combination of these contributions becomes of the order $\Omega^{\frac{4}{1+\eta}}$. There is also a similar cancellation in the collision interal so that the transport scattering rate becomes of the order $\Omega^{\frac{4}{1+\eta}}$. As a result, the smooth fluctuations show no anomalous behavior expected from the singular self-energy correction. On the other hand, for the rough fluctuations $(l > l_c)$, the Landau-interaction part becomes very small and the self-energy part, which is proportional to $\Omega^{\frac{2}{1+\eta}}$, dominates. Also the collision integral becomes of the order $\Omega^{\frac{2}{1+\eta}}$. Therefore, the rough fluctuations show anomalous behavior of the self-energy correction and suggest that the effective mass shows a divergent behavior $m^* \propto \Omega^{-\frac{\eta-1}{\eta+1}}$ for $1 < \eta \leq 2$ and $m^* \propto |\ln \Omega|$ for $\eta = 1$.

From these results, one can understand the density-density and the current-current correlation functions calculated in the perturbation theory [13, 14], which show no anomalous behavior in the long wave length and the low frequency limits. Using the QBE, one can evaluate these correlation functions by taking the angular average of the density or current disturbance due to the external potential and calculating the linear response. Thus, in these correlation functions, the small angular momentum modes are dominating so that the results do not show any singular behavior. Note that the cancellation which exists in the collision integral implies that the transport life time is sufficiently long to explain the long mean free path of the composite fermions in the recent magnetic focusing experiment [10]. For the $2k_F$ response functions, there is no corresponding cancellation between the self-energy part and the Landau-interaction part so that it shows the singular behavior [14].

The QBE in the presence of the small effective magnetic field ΔB was used to understand the energy gap $E_{\rm g}$ of the system. As the case of $\Delta B = 0$, there can be two different behaviors of the generalized Fermi surface displacement. For the smooth fluctuations ($l < l_c \propto E_{\rm g}^{-\frac{1}{1+\eta}}$), the frequency of revolution of the wave packet is given by $\Delta\omega_c = \Delta B/m$, *i.e.*, there is no anomalous behavior after the cancellation between the self-energy and the Landau-interaction parts. For the rough fluctuations, the self-energy part dominates and the frequency of revolution of the wave packet is renormalized as $\Delta\omega_c^* \propto \Delta\omega_c \, E_{\rm g}^{-\frac{\eta-1}{\eta+1}}$. The energy gap can be obtained by quantizing the motion of the wave packet and taking the lowest quantized frequency which is nothing but $\Delta\omega_c^*$. Solving the self-consistent equation $E_{\rm g} = \Delta\omega_c^*$, we get $E_{\rm g} \propto |\Delta B|^{\frac{1+\eta}{2}}$ for $1 < \eta \leq 2$ and $E_{\rm g} \propto \frac{|\Delta B|}{|\ln \Delta B|}$ for $\eta = 1$. These are consistent with the previous results [6, 13, 16].

The excitations of the system were studied from the QBE of the generalized Fermi surface displacement. For $\Delta B = 0$, in the absence of the collision integral, there are two types of the excitations which can be described most easily in the $\Omega - q$ plane. There are particle-hole excitations which exist below an edge $\Omega \propto q^{\frac{1+\eta}{2}}$ $(1 < \eta \leq 2)$ or $\Omega \propto q/|\ln q|$ $(\eta = 1)$. There are also collective modes which exist between $\Omega \propto q^{\frac{1+\eta}{2}}$ $(1 < \eta \leq 2)$, $\Omega \propto q/|\ln q|$ $(\eta = 1)$ and $\Omega \sim v_F q$. However, the distinction between these two different elementary excitations is obscured by the presence of the collision integral which provides the life time of the excitations. In the case of $\Delta B \neq 0$, the QBE in the presence of the finite ΔB is again used to understand the low lying plasma spectrum of the system as a function of q. For $q < q_c$, where $q_c \propto \sqrt{|\Delta B|}$ for $1 < \eta \leq 2$ and $q_c \propto \sqrt{|\Delta B| \ln |\Delta B|}$ for $\eta = 1$, the plasma mode corresponds to a smooth fluctuation of the Fermi surface, and the excitation gap is given by $\Delta\omega_c \sim \Delta B/m$. On the other hand, for $q > q_c$, the plasma mode corresponds to a rough fluctuation of the Fermi surface. As a consequence, the excitation gap becomes much smaller and proportional to $|\Delta B|^{\frac{1+\eta}{2}}$ for $1 < \eta \leq 2$ and $|\Delta B|/|\ln \Delta B|$ for $\eta = 1$. Thus, the lowest excitation spectrum of the system looks like the one given by Figure 4-5, which is consistent with the previous numerical calculations [37].

Applying the picture developed in this paper for the $\nu = 1/2$ metallic state to the magnetic focusing experiment of Ref.[10], we find that the observed oscillations should not be interpreted as the effects due to the focusing of the quarsiparticles. This is because the inelastic mean free path $L_q = v_F^* \tau$ and the life time $\tau \sim 1/T$ of the quasiparticle is quite short. Here v_F^* is the renormalized Fermi velocity of the quasiparticle. For the Coulomb interaction, we find

$$L_q \sim \frac{\sqrt{4\pi n}}{mT\ln(E_F/T)}$$

Here n is the density of the electron, T the temperature, m the bare mass of the composite fermion, and $E_F = \frac{k_F^2}{2m} = \frac{2\pi n}{m}$. Taking $n = 10^{11}cm^{-2}$ and m to be the electron mass in the vacuum (see Ref.[2], Ref.[3] and Ref.[10]), we have

$$L_q \sim 0.26\frac{100\text{mK}}{T}\mu\text{m}$$

At $T = 35$mK, $L_q \sim 0.7\mu$m which is much less than the length of the semi-circular path, 6μm, which connects the two slits. Therefore, the oscillations observed in Ref.[10] cannot be explained by the focusing of the quasiparticles which have a divergent effective mass and a short life time.

There is another way to explain the observed oscillations in Ref.[10]. We can inject a net current into one slit, which causes a dipolar distortion of the local Fermi surface near the slit. The current and the associated dipolar distortion propagate in space according to the QBE and are bended by the effective magnetic field ΔB. This causes the oscillation in the current received by the other slit. According to this picture, the oscillations observed in Ref.[10] is caused by the smooth fluctuations of the Fermi surface whose dynamics is identical to those of a Fermi liquid with a *finite* effective mass. Thus, the oscillations in the magnetic focusing experiments behave as if they are caused by quasiparticles with a finite effective mass and a long life time. The relexation time for the current distribution is given by $\tau_j \sim \frac{E_F}{T^2 \ln (E_F/T)}$ for the Coulomb interaction. This leads to a diffussion length (caused by the gauge fluctuations) $L_j = v_F\tau_j$, where v_F is the bare Fermi velocity of the composite fermions. We find

$$L_j \sim 14\left(\frac{100\text{mK}}{T}\right)^2 \mu\text{m} \tag{4.70}$$

The real diffussion length should be shorter than the above value due to other possible scattering mechanisms. Thus, we expect that the crossover temperature, above which the oscillations disappear, should be lower than 150mK. In the experiment [10], no oscillations were observed above 100mK. Another important consequence of our picture is that, if a time-of-flight measurement can be performed by pulsing the incoming current, the time is given by the bare velocity v_F and *not* the quasiparticle velocity v_F^*.

Finally, we make a remark on the surface acoustic wave experiment. The condition that we can see the resonance between the cyclotron radius and sound wave length is given by $\omega_{\text{cyc}} \gg \omega_{\text{s}}$, where ω_{cyc} is the cyclotron frequency and ω_{s} is the sound wave frequency. The reason is that we can regard the sound wave as a standing wave only when $\omega_{\text{cyc}} \gg \omega_{\text{s}}$. Let us imagine that we are changing ω_{s} such that $\omega_{\text{s}} \approx \Delta\omega_c^*$. If we use the quasiparticle picture to explain the above resonance, then the cyclotron frequency $\omega_{\rm cyc}$ is determined by the divergent effective mass, and $\omega_{\rm cyc}$ should be comparable to $\Delta\omega_c^*$. Therefore, there should not be any resonance because $\omega_{\rm cyc} \approx \omega_{\rm s}$ in this case. However, in reality, the resonance is governed by the smooth fluctuation of the Fermi surface, so that $\omega_{\rm cyc} \approx \Delta\omega_c$ is a cyclotron frequency determined by the finite bare mass of the composite fermion. As a result, one should still see the resonance because $\omega_{\rm cyc} \gg \omega_{\rm s} \approx \Delta\omega_c^*$. Therefore, one can expect that there should be still resonance effects even when the phonon energy exceeds the energy gap determined from the Shubnikov-de Haas oscillations. The bottom line is that the cyclotron frequency measured in acoustic wave experiments can be much larger than the energy gap measured in transport experiments. In a recent experiment of Willet *et. al* [40], resonance was observed when ω_s is larger than the energy gap of the system determined by the large effective mass obtained from the Shubnikov-de Haas oscillations [3]. The authors claimed that this is an apparent contradiction between the surface acoustic wave experiment and the Shubnikov-de Haas oscillations. We would like to point out that the cyclotron frequency (for small q) is determined by the bare mass (In a crude estimation [6], the bare mass is about 1/3 of the electron mass in vacuum). On the other hand, the mass obtained from Shubunikov-de Haas oscillations or from the activation gap in transport measurements is in principle a different mass, which in practice turns out to be of order of the electron mass in vacuum even away from $\nu = 1/2$. Even though we do not understand quantitatively the mass difference, there is in principle no contradiction. The surface acoustic experiment is in fact an excellent way of measuring the bare mass.

Appendix A

Irrelevant contributions to the compressibility

In this appendix, we show that $\frac{\partial n_{a2}}{\partial \mu}$ and $\frac{\partial n_b}{\partial \mu}$ are exponentially smaller than $\frac{\partial n_{a1}}{\partial \mu}$ which is calculated in the main text. As discussed in section 3.4, there is a partial cancellation between Ω_{b1} and Ω_{b2} in Eq.(3-17) due to the f-sum rule given by Eq.(3-32). As a result, Ω_b can be rewritten as

$$\begin{aligned}
\Omega_b &\approx \sum_{\mathbf{q}} \int_0^\infty \frac{dx}{\pi} \, (1+2n_B(x)) \, \widetilde{D}''_{11}(\mathbf{q},x) \left[-\sum_{lm} |M_{lm}(\mathbf{q})|^2 \right. \\
&\quad \times \left. \left(\frac{n_F(\xi_l)-n_F(\xi_m)}{x-\xi_m+\xi_l} - \frac{n_F(\xi_l)-n_F(\xi_m)}{\xi_l-\xi_m} \right) \right] \\
&= \sum_{\mathbf{q}} \int_0^\infty \frac{dx}{\pi} \, (1+2n_B(x)) \, x \, \widetilde{D}''_{11}(\mathbf{q},x) \\
&\quad \times \left[-\sum_{lm} |M_{lm}(\mathbf{q})|^2 \, \frac{n_F(\xi_l)-n_F(\xi_m)}{(x-\xi_m+\xi_l)\,(\xi_l-\xi_m)} \right] . \qquad \text{(A.1)}
\end{aligned}$$

From Eq.(3-16) and Eq. (A.1), we get the lowest order correction to the density of the fermions $n_1 = n_a + n_b$ as follows.

$$\begin{aligned}
n_a &= n_{a1} + n_{a2} \, , \\
n_{a1} &= -\frac{1}{T} \sum_{\mathbf{q}} \sum_{l} |M_{ll}(\mathbf{q})|^2 \, D'_{11}(\mathbf{q},0) \, n_F(\xi_l)(1-n_F(\xi_l))(1-2n_F(\xi_l)) \, , \\
n_{a2} &= -\frac{1}{T} \sum_{\mathbf{q}} \sum_{l\neq m} |M_{lm}(\mathbf{q})|^2 \, D'_{11}(\mathbf{q},\xi_m-\xi_l) \, \left[\, n_F(\xi_m)(1-n_F(\xi_m))(1-n_F(\xi_l)) \right. \\
&\quad \left. - \, n_F(\xi_m) n_F(\xi_l))(1-n_F(\xi_l)) \, \right] \, , \qquad \text{(A.2)}
\end{aligned}$$

and

$$\begin{aligned}
n_b &\approx \frac{1}{T} \sum_{\mathbf{q}} \int_0^\infty \frac{dx}{\pi} \, (1+2n_B(x)) \, x \, D''_{11}(\mathbf{q},x) \\
&\quad \times \left[\sum_{lm} |M_{lm}(\mathbf{q})|^2 \, \frac{n_F(\xi_l)(1-n_F(\xi_l)) - n_F(\xi_m)(1-n_F(\xi_m))}{(x-\xi_m+\xi_l)\,(\xi_l-\xi_m)} \right] . \qquad \text{(A.3)}
\end{aligned}$$

These equations are equivalent to Eq.(3-12).

As shown in Eq.(3-13), in order to calculate the compressibility, one should take the derivative of both $D_{11}(\mathbf{q},x)$ and $\Pi_{11}(\mathbf{q},x)$. Note that $\frac{\partial D_{11}}{\partial\mu} \sim D_{11}^{-2}\frac{\partial \Pi_{11}}{\partial\mu}$ and $\frac{\partial\Pi_{11}}{\partial\mu}$ contains the factor $n_F(\xi_l)(1-n_F(\xi_l))$. Thus $\frac{\partial D_{11}}{\partial\mu}$ generates additional factors $e^{-|\xi_p|/T}$ and $e^{-\xi_{p+1}/T}$. Since we want to keep only the terms which are proportional to $e^{-|\xi_p|/T}$ or $e^{-\xi_{p+1}/T}$, we can ignore the terms $\frac{\partial\Pi_{11}}{\partial\mu}\frac{\partial D_{11}}{\partial\mu}$, which are of order $e^{-2|\xi_p|/T}$. Ignoring these terms in Eq.(3-13) which is equivalent to keeping only the μ dependence in n_F in Eq. (A.2) and Eq. (A.3), the lowest order correction to the compressibility $\frac{\partial n_1}{\partial\mu} = \frac{\partial n_a}{\partial\mu} + \frac{\partial n_b}{\partial\mu}$ can be calculated as follows.

$$
\begin{aligned}
\frac{\partial n_a}{\partial\mu} &= \frac{\partial n_{a1}}{\partial\mu} + \frac{\partial n_{a2}}{\partial\mu}\ , \\
\frac{\partial n_{a1}}{\partial\mu} &\approx -\frac{1}{T^2}\sum_{\mathbf{q}}\sum_{l} |M_{ll}(\mathbf{q})|^2\ D'_{11}(\mathbf{q},0) \\
&\quad \times\ n_F(\xi_l)(1-n_F(\xi_l))\,[\ 1-6\ n_F(\xi_l)(1-n_F(\xi_l))\ ]\ , \\
\frac{\partial n_{a2}}{\partial\mu} &\approx -\frac{1}{T^2}\sum_{\mathbf{q}}\sum_{l\neq m} |M_{lm}(\mathbf{q})|^2\ D'_{11}(\mathbf{q},\xi_m-\xi_l) \\
&\quad \times [\ n_F(\xi_m)(1-n_F(\xi_m))(1-2n_F(\xi_m))(1-n_F(\xi_l)) \\
&\quad\ -\ n_F(\xi_m)n_F(\xi_l)(1-n_F(\xi_l))(1-2n_F(\xi_l))\ ]\ ,
\end{aligned} \tag{A.4}
$$

and

$$
\begin{aligned}
\frac{\partial n_b}{\partial\mu} &\approx -\frac{1}{T^2}\sum_{\mathbf{q}}\int_0^\infty \frac{dx}{\pi}\ (1+2n_B(x))\ x\ D''_{11}(\mathbf{q},x)\Bigg[\ \sum_{lm}\frac{|M_{lm}(\mathbf{q})|^2}{(x-\xi_m+\xi_l)\ (\xi_l-\xi_m)} \\
&\quad \times\Bigg[\ n_F(\xi_l)(1-n_F(\xi_l))(1-2n_F(\xi_l)) \\
&\quad -n_F(\xi_m)(1-n_F(\xi_m))(1-2n_F(\xi_m))\ \Bigg]\Bigg]\ .
\end{aligned} \tag{A.5}
$$

Keeping only the terms that are proportional to $e^{-|\xi_p|/T}$ or $e^{-\xi_{p+1}/T}$, one can show that the contributions from $\frac{\partial n_{a2}}{\partial\mu}$ and $\frac{\partial n_b}{\partial\mu}$ do not contain such terms that are proportional to $e^{-|\xi_p|/T}$ or $e^{-\xi_{p+1}/T}$. This result can be obtained as follows. In each case of $\frac{\partial n_{a2}}{\partial\mu}$ and $\frac{\partial n_b}{\partial\mu}$, the first term and the second term inside the square bracket contain contributions proportional to $e^{-|\xi_p|/T}$ or $e^{-\xi_{p+1}/T}$. It can be seen that these contributions in the first term cancel each other when the chemical potential lies exactly at the middle of the successive effective Landau levels, and thus they correspond to a uniform shift in these Landau levels. The same story applies to the second term in the square bracket. However, it turns out that the contributions from the first term and the second term cancel again each other so that the contributions proportional to $e^{-|\xi_p|/T}$ or $e^{-\xi_{p+1}/T}$ do not exist in general. Thus $\frac{\partial n_{a2}}{\partial\mu} = \mathcal{O}(e^{-2|\xi_p|/T})$ and $\frac{\partial n_b}{\partial\mu} = \mathcal{O}(e^{-2|\xi_p|/T})$ so that we can ignore these contributions compared to $e^{-|\xi_p|/T}$ or $e^{-\xi_{p+1}/T}$.

Appendix B

Quantum Boltzmann equation at finite temperatures

In this appendix, we consider the QBE at finite temperatures. Recall that Im $\Sigma^R(\mathbf{p},\omega)$ at equilibrium diverges at finite temperatures, which has no cutoff [11]. In this case, it is clear from Eq.(4-5) that $G_0^<(\mathbf{p},\omega) = if_0(\omega)A(\mathbf{p},\omega)$ is not well defined. Thus, it is also difficult to define $G^<(\mathbf{p},\omega;\mathbf{r},t)$ for the non-equilibrium case. Since the divergent contribution to the self-energy comes from the gauge field fluctuations with $\nu < T$, where ν is the energy transfer by the gauge field [11], it is worthwhile to separate the gauge field fluctuations into two parts, *i.e.*, $\mathbf{a}(\mathbf{q},\nu) \equiv \mathbf{a}_-(\mathbf{q},\nu)$ for $\nu < T$ and $\mathbf{a}(\mathbf{q},\nu) \equiv \mathbf{a}_+(\mathbf{q},\nu)$ for $\nu > T$, and examine the effects of $\mathbf{a}_+$, $\mathbf{a}_-$ separately.

The classical fluctuation $\mathbf{a}_-$ of the gauge field can be regarded as a vector potential which corresponds to a static but spatially varying magnetic field $\mathbf{b}_- = \nabla \times \mathbf{a}_-$. For a given random 'magnetic' field, $\mathbf{b}_-(\mathbf{r})$, and in a fixed gauge, the fluctuation of the gauge potential $\mathbf{a}_-$ can be very large. The gauge potential can have huge differences from one point to another, as long as the two points are well separated. We know that locally the center of the Fermi surface is at the momentum $\mathbf{p} - \mathbf{a}_-(\mathbf{r})$ around the point $\mathbf{r}$ in space. The huge fluctuation of $\mathbf{a}_-$ indicates that the local Fermi surfaces at different points in space may appear in very different regions in the momentum space. This is the reason why the one-particle Green's function in the *momentum space* is not well defined. This also suggests that the Fermion distribution in the momentum space, $f(\mathbf{p},\omega)$, may be ill-defined. Note that the local Fermi surface can be determined in terms of the velocity of the fermions (*i.e.*, the states with $\frac{m}{2}\mathbf{v}^2 = \frac{1}{2m}(\mathbf{p}-\mathbf{a}_-)^2 < E_F$ are filled) and the velocity is a gauge-invarint physical quantity. This suggests that it is more reasonable to study the fermion distribution in the physical *velocity space.* The above discussion leads us to consider the one-particle Green's function $\tilde{G}(\mathbf{P}_-,\omega;\mathbf{r},t)$ as a function of a new variable $\mathbf{P}_- = m\mathbf{v} = \mathbf{p} - \mathbf{a}_-$. Note that this transformation is reminicent of the procedure we used in the case of the finite effective magnetic field (see section 4.5). We may follow the similar line of derivation to obtain the QBE in the random magnetic field. Since we effectively separate out $\mathbf{a}_-$ fluctuations, the self-energy, which appears in the equation of motion given by Eq.(4-14), should contain only $\mathbf{a}_+$ fluctuations. Therefore, the equation of motion for $\delta G^<(\mathbf{P}_-,\omega;\mathbf{r},t)$ is given by the Fourier transform of Eq.(4-20) with the following replacement. In the first place, the variable $\mathbf{p}$ should be changed to a new variable $\mathbf{P}_- = \mathbf{p} - \mathbf{a}_-$. Secondly, the self-energy $\tilde{\Sigma}$ should be changed to $\tilde{\Sigma}_+$ which contains now only $\mathbf{a}_+$ fluctuations. Finally, as we can see from the case of the finite effective magnetic field in section 4-5, the following term should be added:

$$\frac{\mathbf{P}_-}{m} \cdot \mathbf{b}_-(\mathbf{r}) \times \frac{\partial}{\partial \mathbf{P}_-} \delta G^<(\mathbf{P}_-, \omega; \mathbf{r}, t) \ . \tag{B.1}$$

Note that the equation of motion contains the term which depends on $\mathbf{b}_-$, but does not contain the terms which depend on $\mathbf{a}_-$ in an explicit way. Since we removed the source of the divergence (non-gauge-invariance with respect to $\mathbf{a}_-$), the Green's function $\tilde{G}(\mathbf{P}_-, \omega; \mathbf{r}, t)$ or the corresponding self-energy is now finite for finite T or ω.

Now one can perform the integration $\int d\xi_{\mathbf{P}_-}/2\pi$ of $\delta G^<(\mathbf{P}_-, \omega; \mathbf{r}, t)$ safely to define

$$\begin{aligned} \int \frac{d\xi_{\mathbf{P}_-}}{2\pi} [\, -iG^<(\mathbf{P}_-, \omega; \mathbf{r}, t) \,] \ &\equiv \ f(\theta, \omega; \mathbf{r}, t) \ , \\ \int \frac{d\xi_{\mathbf{P}_-}}{2\pi} [\, iG^>(\mathbf{P}_-, \omega; \mathbf{r}, t) \,] \ &\equiv \ 1 - f(\theta, \omega; \mathbf{r}, t) \ , \end{aligned} \tag{B.2}$$

where θ is the angle between $\mathbf{P}_-$ and a given direction. For a while, let us ignore the contribution coming from the term that depends on $\mathbf{b}_-(\mathbf{r})$ in the equation of motion for $\delta f(\theta, \omega; \mathbf{r}, t)$, which is given by

$$\frac{b_-(\mathbf{r})}{m} \frac{\partial}{\partial \theta} \delta f(\theta, \omega; \mathbf{r}, t) \ . \tag{B.3}$$

In the absence of this term, the equation of motion of the generalized distribution function $\delta f(\theta, \omega; \mathbf{q}, \Omega)$ is given by Eq.(4-34) with the constraint that the lower cuoff T should be introduced in the frequency integrals, which is due to the fact that only $\mathbf{a}_+$ fluctuations should be included. Using the same procedure we used in section 4-4, we can construct the equation of motion for the generalized Fermi surface displacement (in the velocity space) $u(\theta, \mathbf{q}, \Omega) = \int \frac{d\omega}{2\pi} \delta f(\theta, \omega; \mathbf{q}, \omega)$. The corresponding equation is given by Eq.(4-41) with the change that θ_c in the definition of the Landau-interaction-function $F_{\text{Landau}}(\theta)$ is now given by $\theta_c = \frac{1}{k_F} \left(\frac{\gamma \text{Max}(\Omega, T)}{\chi} \right)^{\frac{1}{1+\eta}}$. Therefore, the same arguments for the small and large angular momentum modes can be used to discuss the physical consequences of the QBE and the change is that the crossover angular momentum is now given by $l_c \approx 1/\theta_c \approx k_F \left(\frac{\gamma \text{Max}(\Omega, T)}{\chi} \right)^{-\frac{1}{1+\eta}}$.

Now let us discuss the effect of the term which depends on $b_-(\mathbf{r})$. After integration $\int d\omega/2\pi$ of the QBE for the generalized distribution function $\delta f(\theta, \omega; \mathbf{r}, t)$, this term has the following form in the QBE for $u(\theta, \mathbf{r}, t)$:

$$\frac{b_-(\mathbf{r})}{m} \frac{\partial}{\partial \theta} u(\theta, \mathbf{r}, t) \ . \tag{B.4}$$

This term provides the scattering mechanism due to $\mathbf{a}_-$ fluctuations and generates a dispersion of the angle θ. The transport scattering rate $1/\tau_-$ which is due to $\mathbf{a}_-$ fluctuations can be estimated as follows. In order to examine b_- fluctuations, let us first consider

$$
\begin{aligned}
\langle b_-(\mathbf{q}) b_-(-\mathbf{q}) \rangle &= \int_0^T \frac{d\omega}{2\pi} \, [\, n(\omega) + 1 \,] \, q^2 \, \mathrm{Im} \, D_{11}(q, \omega) \\
&\approx \int_0^T \frac{d\omega}{2\pi} \, \frac{T}{\omega} \, q^2 \, \frac{q\omega/\gamma}{\omega^2 + (\chi q^{1+\eta}/\gamma)^2} \\
&\approx q^3/\gamma \quad \text{for} \quad q \le q_0 \, ,
\end{aligned}
\tag{B.5}
$$

where $q_0 = (\gamma T/\chi)^{\frac{1}{1+\eta}}$. Therefore, the typical length scale of $b_-(\mathbf{r})$ fluctuations is given by $l_0 = 1/q_0$. The typical value of $b_-(\mathbf{r})$ over the length scale l_0 can be estimated from $\langle b_-(\mathbf{r}) b_-(\mathbf{r}') \rangle \approx 1/(\gamma l_0^5)$ for $|\mathbf{r} - \mathbf{r}'| \le l_0$ so that typical $b_- \approx 1/\sqrt{\gamma l_0^5}$. The dispersion of the angle $\Delta\theta$ after the fermion travels over the length l_0 can be estimated as $\Delta\theta = (b_-/m)\Delta t \approx 1/(\sqrt{\gamma l_0^5}\; m)\; (l_0/v_F) \approx 1/(k_F l_0)^{3/2}$. Let $l_{\mathrm{M}} = n l_0$ be the mean free path which is defined by the length scale after which the total dispersion of the angle becomes of the order one. The number n can be estimated by requiring that the total dispersion of the angle $\sqrt{n}\; \Delta\theta \approx \sqrt{n}/(k_F l_0)^{3/2}$ becomes of the order one so that $n \approx (k_F l_0)^3$. Thus, $l_{\mathrm{M}} \approx k_F^3 l_0^4$. From $l_{\mathrm{M}} = v_F \tau_-$, the scattering rate due to $\mathbf{a}_-$ fluctuations can be estimated as $1/\tau_- \propto T^{\frac{4}{1+\eta}}$.

Note that $1/\tau_- \propto T^{\frac{4}{1+\eta}}$ is the same order as that of the scattering rate due to $\mathbf{a}_+$ fluctuations in the case of the small angular momentum modes ($l < l_c$). For $l < l_c$, the contribution from the imaginary part of the self-energy Im $\Sigma^R \propto T^{\frac{2}{1+\eta}}$ is canceled by the contribution from the Landau-interaction function so that the resulting scattering rate is proportional to $T^{\frac{4}{1+\eta}}$. In the other limit of large angular momentum modes ($l > l_c$), $1/\tau_-$ can be completely ignored. This is because the self-energy contribution dominates. Since $1/\tau_- < T$ and it is at most the same order as the scattering rate due to $\mathbf{a}_+$ fluctuations even in the case of the small angular momentum modes, ignoring this contribution does not affect the general consequences of the QBE, which are discussed in sections 4-4, 4-5, and 4-6.

Therefore, the QBE for the generalized distribution function at finite temperatures is essentially given by Eq.(4-34) with the lower cutoff T of the frequency integral in the expression of the contributions from the self-energy and the Landau-interaction-function. As a result, the form of the QBE is the same as that of the zero temperature case and the only difference is that the crossover angle θ_c and the crossover angular momentum l_c are now given by $\theta_c \approx \frac{1}{k_F} \left(\frac{\gamma \ \mathrm{Max}(\Omega, T)}{\chi} \right)^{\frac{1}{1+\eta}}$ and $l_c \approx 1/\theta_c \propto [\ \mathrm{Max}(\Omega, T)\ ]^{-\frac{1}{1+\eta}}$ respectively.

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List of Publications

1) Y. B. Kim and X.-G. Wen, "Effects of collective modes on pair-tunneling into superconductors", Phys. Rev. B **48**, 6319 (1993).

2) Y. B. Kim and X.-G. Wen, "Large N renormalization group study of the commensurate dirty boson problem", Phys. Rev. B **49**, 4043 (1994).

3) M. Sigrist and Y. B. Kim, "Test experiments for time-reversal symmetry breaking superconductivity", J. Phys. Soc. Jpn **64**, 4314 (1994).

4) Y. B. Kim and X.-G. Wen, "Instantons and the spectral function of electrons in the half-filled Landau level", Phys. Rev. B **50**, 8078 (1994).

5) Y. B. Kim, A. Furusaki, X.-G. Wen, and P. A. Lee, "Gauge-invariant response functions coupled to a gauge field", Phys. Rev. B **50**, 17917 (1994).

6) Y. B. Kim, P. A. Lee, X.-G. Wen, and P. C. E. Stamp, "Influence of gauge-field fluctuations on composite fermions near the half-filled state", Phys. Rev. B **51**, 10779 (1995).

7) Y. B. Kim, P. A. Lee, and X.-G. Wen, "Quantum Boltzmann equation of composite fermions interacting with a gauge field", submitted to Phys. Rev. B for publication.

8) Y. B. Kim, A. Furusaki, and D. K. K. Lee, "On a network model of localization in a random magnetic field", submitted to Phys. Rev. B for publication.